Central Extensions of Root Graded Lie Algebras

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Abstract. We study the central extensions of Lie algebras graded by an irreducible locally finite root system.
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0. INTRODUCTION

In 1992, S. Berman and R. Moody [4] introduced the notion of a *Lie algebra graded by an irreducible reduced finite root system.* This definition was generalized by E. Neher [6] in 1996 to Lie algebras graded by a reduced locally finite root system and also by B. Allison, G. Benkart and Y. Gao [2] in 2002 to Lie algebras graded by an irreducible finite root system of type BC. Finally this definition was generalized to Lie algebras graded by a locally finite root system (not necessarily reduced) in [6] and [8]. A complete description of Lie algebras graded by an irreducible finite root system is given in [4], [3], [2] and [5]. In [6], E. Neher realizes Lie algebras graded by a reduced locally finite root system other than root systems of types F_4, G_2 and E_8 as central extensions of Tits-Kantor-Koecher algebras of certain Jordan pairs. In [8], the author studies Lie algebras graded by an infinite irreducible locally finite root system (not necessarily reduced) and gives a complete description of these Lie algebras.

Central extensions play a very important role in the theory of Lie algebras. Central extensions of Lie algebras graded by an irreducible finite root system is given in [1], [2] and [5]. The universal central extension of Lie algebras graded by a reduced locally finite root system is studied by A. Walte in her Ph.D. thesis [7] in 2010. In 2011, E. Neher and J. Sun prove that the universal central extension of a direct limit of a class $\{\mathcal{L}_i \mid i \in I\}$ of perfect Lie superalgebras coincides with the direct limit of universal central extension of $\mathcal{L}_i$'s. As a by-product, they determine the universal central extension of Lie algebras graded by an irreducible reduced locally finite root system. Here in this work we study the central extension of a Lie algebra graded by an irreducible locally finite root system (not necessarily reduced). According to [8], if X is the type of an irreducible locally finite root system, for a specific quadruple $\mathfrak{q}$ called a *coordinate quadruple* of type X, one can associate an algebra $\mathfrak{b}(\mathfrak{q})$ and a Lie algebra $\{\mathfrak{b}(\mathfrak{q}), \mathfrak{b}(\mathfrak{q})\}$. The structure of Lie algebras graded by an irreducible locally finite root system R of type X just depends on coordinate quadruples $\mathfrak{q}$ of type X and certain subspaces $\mathcal{K}$ of $\{\mathfrak{b}(\mathfrak{q}), \mathfrak{b}(\mathfrak{q})\}$ said to satisfies the uniform property on $\mathfrak{b}(\mathfrak{q})$. In

fact corresponding to a coordinate quadruple $\mathfrak{q}$ of type X and a subspace $\mathcal{K}$ of $\{\mathfrak{b}(\mathfrak{q}), \mathfrak{b}(\mathfrak{q})\}$ satisfying the uniform property on $\mathfrak{b}(\mathfrak{q})$, the author associates a Lie algebra $\mathcal{L}(\mathfrak{q}, \mathcal{K})$ and shows that it is a Lie algebra graded by the irreducible locally finite root system R of type X. Conversely she proves that any $R-$graded Lie algebras is isomorphic to such a Lie algebra. In this work we study the central extensions of root graded Lie algebras. We prove that a perfect central extension of a Lie algebra graded by an irreducible locally finite root system R is a Lie algebra graded by the same root system R with the same coordinate quadruple. Moreover, we prove that the universal central extension of a Lie algebra $\mathcal{L} = \mathcal{L}(\mathfrak{q}, \mathcal{K})$ graded by an irreducible locally finite root system R is $\mathcal{L}(\mathfrak{q}, \{0\})$.

1. Preliminary

By a *star algebra* $(\mathfrak{A}, \star)$, we mean an algebra $\mathfrak{A}$ together with a self-inverting antiautomorphism $\star$ which is referred to as an *involution*.

We call a quadruple $(\mathfrak{a}, *, \mathcal{C}, f)$, a *coordinate quadruple* if one of the followings holds:

- (Type A) $\mathfrak{a}$ is a unital associative algebra, $* = id_{\mathfrak{a}}$, $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.

- (Type B) $\mathfrak{a} = \mathcal{A} \oplus \mathcal{B}$ where $\mathcal{A}$ is a unital commutative associative algebra and $\mathcal{B}$ is a unital associative $\mathcal{A}$-module equipped with a symmetric bilinear form and $\mathfrak{a}$ is the corresponding Clifford Jordan algebra, $*$ is a linear transformation fixing the elements of $\mathcal{A}$ and skew fixing the elements of $\mathcal{B}$, $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.

- (Type C) $\mathfrak{a}$ is a unital associative algebra, $*$ is an involution on $\mathfrak{a}$, $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.

- (Type D) $\mathfrak{a}$ is a unital commutative associative algebra $* = id_{\mathfrak{a}}$, $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.

- (Type BC) $\mathfrak{a}$ is a unital associative algebra, $*$ is an involution on $\mathfrak{a}$, $\mathcal{C}$ is a unital associative $\mathfrak{a}$-module and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is a skew-hermitian form.

Suppose that $\mathfrak{q} := (\mathfrak{a}, *, \mathcal{C}, f)$ is a coordinate quadruple. Denote by $\mathcal{A}$ and $\mathcal{B}$, the fixed and the skew fixed points of $\mathfrak{a}$ under $*$, respectively. Set $\mathfrak{b} := \mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f) := \mathfrak{a} \oplus \mathcal{C}$ and define

$$(1.1) \qquad \begin{aligned} \cdot : \mathfrak{b} \times \mathfrak{b} &\longrightarrow \mathfrak{b} \\ (\alpha_1 + c_1, \alpha_2 + c_2) &\mapsto (\alpha_1 \cdot \alpha_2) + f(c_1, c_2) + \alpha_1 \cdot c_2 + \alpha_2^* \cdot c_1, \end{aligned}$$

for $\alpha_1, \alpha_2 \in \mathfrak{a}$ and $c_1, c_2 \in \mathcal{C}$. Also for $\beta, \beta' \in \mathfrak{b}$, set

$$(1.2) \qquad \beta \circ \beta' := \beta \cdot \beta' + \beta' \cdot \beta \quad \text{and} \quad [\beta, \beta'] := \beta \cdot \beta' - \beta' \cdot \beta,$$

and for $c, c' \in \mathcal{C}$, define

(1.3)
$$\diamond : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{A}, \quad (c, c') \mapsto \tfrac{f(c,c')-f(c',c)}{2}; \ c, c' \in \mathcal{C},$$
$$\heartsuit : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{B}, \quad (c, c') \mapsto \tfrac{f(c,c')+f(c',c)}{2}; \ c, c' \in \mathcal{C}.$$

Now suppose that ℓ is a positive integer and for $\alpha, \alpha' \in \mathfrak{a}$ and $c, c' \in \mathcal{C}$, consider the following endomorphisms

(1.4)

$d_{\alpha,\alpha'} : \mathfrak{b} \longrightarrow \mathfrak{b},$

$$\beta \mapsto \begin{cases} \frac{1}{\ell+1}[[\alpha, \alpha'], \beta] & \mathfrak{q} \text{ is of type } A, \ \beta \in \mathfrak{b}, \\ \alpha'(\alpha\beta) - \alpha(\alpha'\beta) & \mathfrak{q} \text{ is of type } B, \ \beta \in \mathfrak{b}, \\ \frac{1}{4\ell}[[\alpha, \alpha'] + [\alpha^*, \alpha'^*], \beta] & \mathfrak{q} \text{ is of type } C \text{ or } BC, \ \beta \in \mathfrak{a}, \\ \frac{1}{4\ell}([\alpha, \alpha'] + [\alpha^*, \alpha'^*]) \cdot \beta & \mathfrak{q} \text{ is of type } C \text{ or } BC, \ \beta \in \mathcal{C}, \\ 0 & \mathfrak{q} \text{ is of type } D, \ \beta \in \mathfrak{b}, \end{cases}$$

$d_{c,c'} : \mathfrak{b} \longrightarrow \mathfrak{b},$

$$\beta \mapsto \begin{cases} \frac{-1}{2\ell}[c \heartsuit c', \beta] & \mathfrak{q} \text{ is of type } BC, \ \beta \in \mathfrak{a}, \\ \frac{-1}{2\ell}(c \heartsuit c') \cdot \beta - \frac{1}{2}(f(\beta, c') \cdot c + f(\beta, c) \cdot c') & \mathfrak{q} \text{ is of type } BC, \ \beta \in \mathcal{C}, \\ 0 & \text{otherwise}, \end{cases}$$

$d_{\alpha,c} := d_{c,\alpha} := 0,$

$d_{\alpha+c,\alpha'+c'} := d_{\alpha,\alpha'} + d_{c,c'}.$

One can see that for $\beta, \beta' \in \mathfrak{b}$, $d_{\beta,\beta'} \in Der(\mathfrak{b})$. Next take K to be a subspace of $\mathfrak{b} \otimes \mathfrak{b}$ spanned by

$$\alpha \otimes c, \ c \otimes \alpha, \ a \otimes b,$$
$$\alpha \otimes \alpha' + \alpha' \otimes \alpha, \ c \otimes c' - c' \otimes c,$$
$$(\alpha \cdot \alpha') \otimes \alpha'' + (\alpha'' \cdot \alpha) \otimes \alpha' + (\alpha' \cdot \alpha'') \otimes \alpha,$$
$$f(c, c') \otimes \alpha + (\alpha^* \cdot c') \otimes c - (\alpha \cdot c) \otimes c'$$

for $\alpha, \alpha', \alpha'' \in \mathfrak{a}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $c, c' \in \mathcal{C}$. Then $(\mathfrak{b} \otimes \mathfrak{b})/K$ is a Lie algebra under the following Lie bracket

(1.5)
$$[(\beta_1 \otimes \beta_2) + K, (\beta_1' \otimes \beta_2') + K] := ((d_{\beta_1,\beta_2}(\beta_1') \otimes \beta_2') + K) + (\beta_1' \otimes d_{\beta_1,\beta_2}(\beta_2')) + K)$$

for $\beta_1, \beta_2, \beta_1', \beta_2' \in \mathfrak{b}$ (see [2, Proposition 5.23] and [1]). We denote this Lie algebra by $\{\mathfrak{b}, \mathfrak{b}\}$ (or $\{\mathfrak{b}, \mathfrak{b}\}$ if there is no confusion) and for $\beta_1, \beta_2 \in \mathfrak{b}$, we denote $(\beta_1 \otimes \beta_2) + K$ by $\{\beta_1, \beta_2\}$ (or $\{\beta_1, \beta_2\}$ if there is no confusion). We recall the *full skew-dihedral homology group*

$$\mathrm{HF}(\mathfrak{b}) := \{\sum_{i=1}^{n}\{\beta_i, \beta_i'\} \in \{\mathfrak{b}, \mathfrak{b}\} \mid \sum_{i=1}^{n} d_{\beta_i,\beta_i'} = 0\}$$

of $\mathfrak{b}$ (with respect to ℓ) from [2] and [1] and note that it is a subset of the center of $\{\mathfrak{b}, \mathfrak{b}\}$. For $\beta_1 = a_1 + b_1 + c_1 \in \mathfrak{b}$ and $\beta_2 = a_2 + b_2 + c_2 \in \mathfrak{b}$ with

$a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $c_1, c_2 \in \mathcal{C}$, set

$$(1.6) \qquad \beta^*_{\beta_1, \beta_2} := [a_1, a_2] + [b_1, b_2] - c_1 \heartsuit c_2; \quad \beta^*_1 := c_1, \quad \beta^*_2 := c_2.$$

We say a subset $\mathcal{K}$ of the full skew-dihedral homology group of $\mathfrak{b}$ satisfies the "*uniform property on* $\mathfrak{b}$" if for $\beta_1, \beta'_1, \ldots, \beta_n, \beta'_n \in \mathfrak{b}$, $\sum_{i=1}^{n} \{\beta_i, \beta'_i\} \in \mathcal{K}$ implies that $\sum_{i=1}^{n} \beta^*_{\beta_i, \beta'_i} = 0$.

Remark 1.7. We point it out that if $\mathfrak{q}$ is a coordinate quadruple and $\mathrm{HF}(\mathfrak{b}(\mathfrak{q}))$ has a subspace satisfying the uniform property on $\mathfrak{b}(\mathfrak{q})$, then $\{0\}$ also satisfies the uniform property on $\mathfrak{b}$.

Suppose that I is a nonempty index set and set $J := I \uplus \bar{I}$. Suppose that $\mathcal{V}$ is a vector space with a fixed basis $\{v_j \mid j \in J\}$. One knows that $\mathfrak{gl}(\mathcal{V}) := \mathrm{End}(\mathcal{V})$ together with

$$[\cdot, \cdot] : \mathfrak{gl}(\mathcal{V}) \times \mathfrak{gl}(\mathcal{V}) \longrightarrow \mathfrak{gl}(\mathcal{V}); \ (X, Y) \mapsto XY - YX; \ \ X, Y \in \mathfrak{gl}(\mathcal{V})$$

is a Lie algebra. Now for $j, k \in J$, define

$$(1.8) \qquad\qquad e_{j,k} : \mathcal{V} \longrightarrow \mathcal{V}; \ \ v_i \mapsto \delta_{k,i} v_j, \quad (i \in J),$$

then $\mathfrak{gl}(J) := \mathrm{span}_{\mathbb{F}}\{e_{j,k} \mid j, k \in J\}$ is a Lie subalgebra of $\mathfrak{gl}(\mathcal{V})$. Consider the bilinear form $(\cdot, \cdot)$ on $\mathcal{V}$ defined by

$$(1.9) \quad (v_j, v_{\bar{k}}) := -(v_{\bar{k}}, v_j) := 2\delta_{j,k}, \ (v_j, v_k) := 0, \ (v_{\bar{j}}, v_{\bar{k}}) := 0; \quad (j, k \in I),$$

and set

$$\mathcal{G} := \mathfrak{sp}(I) := \{\phi \in \mathfrak{gl}(J) \mid (\phi(v), w) = -(v, \phi(w)), \ \text{for all} \ v, w \in \mathcal{V}\}.$$

Also for a fixed subset I_0 of I, take $\{I_\lambda \mid \lambda \in \Lambda\}$ to be the class of all finite subsets of I containing I_0, in which Λ is an index set containing 0, and for each $\lambda \in \Lambda$, set

$$(1.10) \qquad\qquad \mathcal{G}^\lambda := \mathcal{G} \cap \mathrm{span}\{e_{r,s} \mid r, s \in I_\lambda \cup \bar{I}_\lambda\}.$$

Then $\mathcal{G}$ is a locally finite split simple Lie subalgebra of $\mathfrak{gl}(J)$ with splitting Cartan subalgebra $\mathcal{H} := \mathrm{span}_{\mathbb{F}}\{h_i := e_{i,i} - e_{\bar{i},\bar{i}} \mid i \in I\}$. Moreover, for $i, j \in I$ with $i \neq j$, we have

$$\mathcal{G}_{\epsilon_i - \epsilon_j} = \mathbb{F}(e_{i,j} - e_{\bar{j},\bar{i}}), \ \mathcal{G}_{\epsilon_i + \epsilon_j} = \mathbb{F}(e_{i,\bar{j}} + e_{j,\bar{i}}), \ \mathcal{G}_{-\epsilon_i - \epsilon_j} = \mathbb{F}(e_{\bar{i},j} + e_{\bar{j},i}),$$
$$\mathcal{G}_{2\epsilon_i} = \mathbb{F}e_{i,\bar{i}}, \ \mathcal{G}_{-2\epsilon_i} = \mathbb{F}e_{\bar{i},i}.$$

Also for $\lambda \in \Lambda$, $\mathcal{G}^\lambda$ is a finite dimensional split simple Lie subalgebra of type C, with splitting Cartan subalgebra $\mathcal{H}^\lambda := \mathcal{H} \cap \mathcal{G}^\lambda$, and $\mathcal{G}$ is the direct union of $\{\mathcal{G}^\lambda \mid \lambda \in \Lambda\}$.

Define

$$\pi_1 : \mathcal{G} \longrightarrow \mathrm{End}(\mathcal{V}); \ \pi(\phi)(v) := \phi(v); \quad \phi \in \mathcal{G}, \ v \in \mathcal{V}.$$

Then π_1 is an irreducible representation of $\mathcal{G}$ in $\mathcal{V}$ equipped with a weight space decomposition with respect to $\mathcal{H}$ whose set of weights is $\{\pm \epsilon_i \mid i \in I\}$ with $\mathcal{V}_{\epsilon_i} = \mathbb{F}v_i$ and $\mathcal{V}_{-\epsilon_i} = \mathbb{F}v_{\bar{i}}$ for $i \in I$. Also for

$$(1.11) \quad \mathcal{S} := \{\phi \in \mathfrak{gl}(J) \mid tr(\phi) = 0, (\phi(v), w) = (v, \phi(w)), \ \text{for all} \ v, w \in \mathcal{V}\},$$

we have that

$$\pi_2 : \mathcal{G} \longrightarrow \mathrm{End}(\mathcal{S}); \ \pi_2(X)(Y) := [X, Y]; \ \ X \in \mathcal{G}, \ Y \in \mathcal{S}$$

is an irreducible representation of $\mathcal{G}$ in $\mathcal{S}$ equipped with a weight space decomposition with respect to $\mathcal{H}$ whose set of weights is $\{0, \pm(\epsilon_i \pm \epsilon_j) \mid i, j \in I, \ i \neq j\}$ with $\mathcal{S}_0 = \mathrm{span}_{\mathbb{F}}\{e_{r,r} + e_{\bar{r},\bar{r}} - \frac{1}{|I_\lambda|} \sum_{i \in I_\lambda} (e_{i,i} + e_{\bar{i},\bar{i}}) \mid \lambda \in \Lambda, r \in I_\lambda\}$, $\mathcal{S}_{\epsilon_i + \epsilon_j} = \mathbb{F}(e_{i,\bar{j}} - e_{j,\bar{i}})$, $\mathcal{S}_{-\epsilon_i - \epsilon_j} = \mathbb{F}(e_{\bar{i},j} - e_{\bar{j},i})$ and $\mathcal{S}_{\epsilon_i - \epsilon_j} = \mathbb{F}(e_{i,j} + e_{\bar{j},\bar{i}})$ $(i, j \in I, i \neq j)$. Next for $\lambda \in \Lambda$, set

(1.12)
$$\begin{aligned} \mathcal{V}^\lambda &:= \mathrm{span}_{\mathbb{F}}\{v_r \mid r \in I_\lambda \cup \bar{I}_\lambda\}, \\ \mathcal{S}^\lambda &:= \mathcal{S} \cap \mathrm{span}_{\mathbb{F}}\{e_{r,s} \mid r, s \in I_\lambda \cup \bar{I}_\lambda\}. \end{aligned}$$

Then $\mathcal{V}^\lambda$ and $\mathcal{S}^\lambda$ are irreducible finite dimensional $\mathcal{G}^\lambda$−modules with the set of weights $(R_\lambda)_{sh}$ and $\{0\} \cup (R_\lambda)_{lg}$ respectively.

Theorem 1.13 (Recognition Theorem for Type BC). *Suppose that I is an infinite index set and ℓ is an integer greater than 3. Assume R is an irreducible locally finite root system of type BC_I and $\mathcal{V}$ is a vector space with a basis $\{v_i \mid i \in I \cup \bar{I}\}$. Suppose that $(\cdot, \cdot)$ is a bilinear form as in (1.9), set $\mathcal{G} := \mathfrak{sp}(I)$ and consider $\mathcal{S}$ as in (1.11). Fix a subset I_0 of I of cardinality ℓ and take R_0 to be the full irreducible subsystem of R of type BC_{I_0}. Suppose that $\{R_\lambda \mid \lambda \in \Lambda\}$ is the class of all finite irreducible full subsystems of R containing R_0, where Λ is an index set containing zero. For $\lambda \in \Lambda$, take $\mathcal{G}^\lambda$ as in Lemma 1.10 and $\mathcal{V}^\lambda, \mathcal{S}^\lambda$ as in (1.12). Next define*

$$\begin{aligned} \mathfrak{I}_\lambda : \mathcal{V} &\longrightarrow \mathcal{V} \\ v_i &\mapsto \begin{cases} v_i & i \in I_\lambda \cup \bar{I}_\lambda \\ 0 & otherwise \end{cases} \end{aligned}$$

and for $e, f \in \mathcal{G} \cup \mathcal{S}$, define

$$e \circ f := ef + fe - \frac{tr(ef)}{l} \mathfrak{I}_0.$$

*(i) Suppose that $(\mathfrak{a}, *, \mathcal{C}, f)$ is a coordinate quadruple of type BC and $\mathcal{A}$, $\mathcal{B}$ are $*$-fixed and $*$-skew fixed points of $\mathfrak{a}$ respectively. Set $\mathfrak{b} := \mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ and take $[\cdot, \cdot], \circ, \heartsuit, \diamond$ to be as in Subsection **??**. For $\beta_1, \beta_2 \in \mathfrak{b}$, consider d_{β_1, β_2} as in (1.4) and take $\beta^*_{\beta_1, \beta_2}, \beta^*_1$ and β^*_2 as in (1.6). For a subset $\mathcal{K}$ of $\mathrm{HF}(\mathfrak{b})$ satisfying the uniform property on $\mathfrak{b}$, set*

$$\mathcal{L}(\mathfrak{b}, \mathcal{K}) := (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus (\{\mathfrak{b}, \mathfrak{b}\}/\mathcal{K}).$$

Then setting $\langle \beta, \beta' \rangle := \{\beta, \beta'\} + \mathcal{K}$, $\beta, \beta' \in \mathfrak{b}$, $\mathcal{L}(\mathfrak{b}, \mathcal{K})$ *together with*
(1.14)

$$[x \otimes a, y \otimes a'] = [x, y] \otimes \tfrac{1}{2}(a \circ a') + (x \circ y) \otimes \tfrac{1}{2}[a, a'] + tr(xy)\langle a, a' \rangle,$$

$$[x \otimes a, s \otimes b] = (x \circ s) \otimes \tfrac{1}{2}[a, b] + [x, s] \otimes \tfrac{1}{2}(a \circ b) = -[s \otimes b, x \otimes a],$$

$$[s \otimes b, t \otimes b'] = [s, t] \otimes \tfrac{1}{2}(b \circ b') + (s \circ t) \otimes \tfrac{1}{2}[b, b'] + tr(st)\langle b, b' \rangle,$$

$$[x \otimes a, u \otimes c] = xu \otimes a \cdot c = -[u \otimes c, x \otimes a],$$

$$[s \otimes b, u \otimes c] = su \otimes b \cdot c = -[u \otimes c, s \otimes b],$$

$$[u \otimes c, v \otimes c'] = (u \circ v) \otimes (c \diamond c') + [u, v] \otimes (c \heartsuit c') + (u, v)\langle c, c' \rangle,$$

$$[\langle \beta_1, \beta_2 \rangle, x \otimes a] = \tfrac{-1}{4\ell}((x \circ \mathfrak{I}_0) \otimes [a, \beta^*_{\beta_1, \beta_2}] + [x, \mathfrak{I}_0] \otimes (a \circ \beta^*_{\beta_1, \beta_2})),$$

$$[\langle \beta_1, \beta_2 \rangle, s \otimes b] = \tfrac{-1}{4\ell}([s, \mathfrak{I}_0] \otimes (b \circ \beta^*_{\beta_1, \beta_2}) + (s \circ \mathfrak{I}_0) \otimes [b, \beta^*_{\beta_1, \beta_2}] + 2tr(s\mathfrak{I}_0)\langle b, \beta^*_{\beta_1, \beta_2} \rangle),$$

$$[\langle \beta_1, \beta_2 \rangle, v \otimes c] = \tfrac{1}{2\ell}\mathfrak{I}_0 v \otimes (\beta^*_{\beta_1, \beta_2} \cdot c) - \tfrac{1}{2} v \otimes (f(c, \beta^*_2) \cdot \beta^*_1 + f(c, \beta^*_1) \cdot \beta^*_2)$$

$$[\langle \beta_1, \beta_2 \rangle, \langle \beta'_1, \beta'_2 \rangle] = \langle d^\ell_{\beta_1, \beta_2}(\beta'_1), \beta'_2 \rangle + \langle \beta'_1, d^\ell_{\beta_1, \beta_2}(\beta'_2) \rangle$$

for $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $u, v \in \mathcal{V}$, $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $c, c' \in \mathcal{C}$, $\beta_1, \beta_2, \beta'_1, \beta'_2 \in \mathfrak{b}$, *is an R-graded Lie algebra with grading pair* $(\mathcal{G}, \mathcal{H})$ *where* $\mathcal{H}$ *is the splitting Cartan subalgebra of* $\mathcal{G}$.

(ii) If $\mathcal{L}$ *is an R-graded Lie algebra with grading pair* $(\mathfrak{g}, \mathfrak{h})$, *then there is a coordinate quadruple* $(\mathfrak{a}, *, \mathcal{C}, f)$ *of type BC and a subspace* $\mathcal{K}$ *of* $\mathfrak{b} := \mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ *satisfying the uniform property on* $\mathfrak{b}$ *such that* $\mathcal{L}$ *is isomorphic to* $\mathcal{L}(\mathfrak{b}, \mathcal{K})$.

A Lie algebra epimorphism $\pi : \tilde{\mathcal{L}} \longrightarrow \mathcal{L}$ from $(\tilde{\mathcal{L}}, [\cdot, \tilde{\cdot}])$ to $(\mathcal{L}, [\cdot, \cdot])$ is called a *central extension* of $\mathcal{L}$ if $C := ker(\pi) \subseteq Z(\tilde{\mathcal{L}})$. One knows that there is a subspace $\mathcal{L}'$ of $\tilde{\mathcal{L}}$ such that $\pi(\mathcal{L}') = \mathcal{L}$, $\pi|_{\mathcal{L}'} : \mathcal{L}' \longrightarrow \mathcal{L}$ is a linear isomorphism and $\tilde{\mathcal{L}} = \mathcal{L}' \oplus ker(\pi)$. For $x \in \tilde{\mathcal{L}}$, take $x' \in \mathcal{L}'$ and $x'' \in ker(\pi)$ to be the image of x under the projection maps of $\mathcal{L}$ on $\mathcal{L}'$ and $ker(\pi)$ respectively, then for $x, y \in \tilde{\mathcal{L}}$, $[x, \tilde{y}] = [x, y]' + [x, y]''$. One can see that $(\mathcal{L}', [\cdot, \cdot]')$ is a Lie algebra and $\pi|_{\mathcal{L}'} : (\mathcal{L}', [\cdot, \cdot]') \longrightarrow (\mathcal{L}, [\cdot, \cdot])$ is a Lie algebra isomorphism. Also $\tau : \mathcal{L}' \times \mathcal{L}' \longrightarrow C$ mapping (x, y) to $[x, y]''$ is a 2$-$cocycle. We identify $\mathcal{L}'$ with $\mathcal{L}$ via π, therefore we have $\tilde{\mathcal{L}} = \mathcal{L} \oplus C$, $\pi : \tilde{\mathcal{L}} \longrightarrow \mathcal{L}$ is the projection map and for $x, y \in \mathcal{L}$, $e, f \in C$, $[x + e, y + \tilde{f}] = [x, y] + \tau(x, y)$. The central extension π is called *perfect* if $\tilde{\mathcal{L}}$ is a perfect Lie algebra.

Lemma 1.15. *Suppose* $\mathcal{L}$ *is a Lie algebra and* $\tau : \mathcal{L} \times \mathcal{L} \longrightarrow C$ *is a* $2-$*cocycle. Consider the corresponding central extension* $\tilde{\mathcal{L}} = \mathcal{L} \oplus C$ *with Lie bracket* $[\cdot, \tilde{\cdot}]$ *as above. Let* $\mathcal{G}$ *be a finite dimensional simple Lie subalgebra of* $\mathcal{L}$ *and consider* $\tilde{\mathcal{L}}$ *as a* $\mathcal{G}-$*module via the action action*

$$\cdot : \mathcal{G} \times \tilde{\mathcal{L}} \longrightarrow \tilde{\mathcal{L}}$$
$$(x, y) \mapsto [x, \tilde{y}]; \quad x \in \mathcal{G}, \ y \in \tilde{\mathcal{L}}.$$

If D *is a trivial* $\mathcal{G}-$*submodule of* $\mathcal{L}$ *via the adjoint representation, then* D *is a trivial* $\mathcal{G}-$*submodule of* $\tilde{\mathcal{L}}$, *in particular* $\tau(\mathcal{G}, D) = \{0\}$.

Proof. Consider the $\mathcal{G}-$submodule $D \oplus \tau(\mathcal{G}, D)$ of $\tilde{\mathcal{L}}$. If $d_1, \ldots, d_n \in D$ and $r_1, \ldots, r_n \in \tau(\mathcal{G}, D)$, then $\{d_1 + r_1, \ldots, d_n + r_n\}$ is a subset of

$\text{span}_{\mathbb{F}}\{d_1, \ldots, d_n\} + \sum_{i=1}^n \tau(\mathcal{G}, d_i) + \text{span}_{\mathbb{F}}\{r_1, \ldots, r_n\}$ which is a finite dimensional $\mathcal{G}-$submodule of $D \oplus \tau(\mathcal{G}, D)$. This means that $D \oplus \tau(\mathcal{G}, D)$ is a locally finite $\mathcal{G}-$module and so it is completely reducible as $\mathcal{G}$ is a finite dimensional simple Lie algebra. Next we note that $\tau(\mathcal{G}, D)$ is a trivial $\mathcal{G}-$submodule of $\mathcal{D} \oplus \tau(\mathcal{G}, \mathcal{D})$, so there is a submodule $\dot{D}$ of $D \oplus \tau(\mathcal{G}, D)$ such that $D \oplus \tau(\mathcal{G}, D) = \dot{D} \oplus \tau(\mathcal{G}, D)$. Now for $\dot{d} \in \dot{D}$, there is $d \in D$ and $r \in \tau(\mathcal{G}, D)$ such that $\dot{d} = d + r$. If $x \in \mathcal{G}$, we have $[x, \dot{d}] = \tau(x, d)$. But $\dot{D}$ is a $\mathcal{G}-$submodule of $\tilde{\mathcal{L}}$, so $[x, \dot{d}] = \tau(x, d) \in \dot{D} \cap \tau(\mathcal{G}, D) = \{0\}$. Therefore $\dot{D}$ is a trivial $\mathcal{G}-$submodule of $\tilde{\mathcal{L}}$ and so $D \oplus \tau(\mathcal{G}, D) = \dot{D} \oplus \tau(\mathcal{G}, D)$ is a trivial $\mathcal{G}-$module. In particular D is a trivial $\mathcal{G}-$submodule of $\tilde{\mathcal{L}}$ and so $\tau(\mathcal{G}, D) = \{0\}$. $\qquad\square$

Lemma 1.16. *Suppose that R is an irreducible locally finite root system and $\mathcal{L} = \oplus_{\alpha \in R} \mathcal{L}_\alpha$ is an $R-$graded Lie algebra with grading pair $(\mathcal{G}, \mathcal{H})$. Suppose that $\tau : \mathcal{L} \times \mathcal{L} \longrightarrow C$ is a $2-$cocycle satisfying $\tau(\mathcal{L}, \mathcal{G}) = \{0\}$. Consider the corresponding central extension $\pi : (\tilde{\mathcal{L}}, [\cdot, \cdot]) \longrightarrow \mathcal{L}$ and suppose $\tilde{\mathcal{L}}$ is perfect, then $\tilde{\mathcal{L}} = \oplus_{\alpha \in R} \tilde{\mathcal{L}}_\alpha$ with*

$$(1.17) \qquad \tilde{\mathcal{L}}_\alpha := \begin{cases} \mathcal{L}_\alpha & \text{if } \alpha \in R \setminus \{0\} \\ \mathcal{L}_0 \oplus C & \text{if } \alpha = 0 \end{cases}$$

is an $R-$graded Lie algebra with grading pair $(\mathcal{G}, \mathcal{H})$. Moreover, if R is a finite root system, then the coordinate quadruple of $\mathcal{L}$ coincides with the coordinate quadruple of $\tilde{\mathcal{L}}$.

Proof. We know $\tilde{\mathcal{L}} = \mathcal{L} \oplus \ker(\pi)$ and that the corresponding $2-$cocycle τ satisfies $\tau(\mathcal{L}, \mathcal{G}) = \{0\}$. Since $\tau(\mathcal{L}, \mathcal{G}) = \{0\}$, we get that $\mathcal{G}$ is a subalgebra of $\tilde{\mathcal{L}}$ and that (1.17) defines a weight space decomposition for $\tilde{\mathcal{L}}$ with respect to $\mathcal{H}$. So to complete the proof, it is enough to show that $\tilde{\mathcal{L}}_0 = \sum_{\alpha \in R \setminus \{0\}} [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_{-\alpha}]$. For this, we note that $[\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_\beta] \subseteq \tilde{\mathcal{L}}_{\alpha+\beta}$ for $\alpha, \beta \in R$ and so

$$\begin{aligned} \tilde{\mathcal{L}}_0 \subseteq \tilde{\mathcal{L}} = [\tilde{\mathcal{L}}, \tilde{\mathcal{L}}] \quad &= \quad \sum_{\alpha, \beta \in R} [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_\beta] \\ &= \quad \sum_{\alpha, \beta; \alpha+\beta=0} [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_\beta] + \sum_{\alpha, \beta; \alpha+\beta\neq0} [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_\beta] \\ &\subseteq \quad \sum_{\alpha, \beta; \alpha+\beta=0} [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_\beta] + \sum_{\alpha, \beta; \alpha+\beta\neq0} \tilde{\mathcal{L}}_{\alpha+\beta}. \end{aligned}$$

Now as $\sum_{\alpha \in R} \tilde{\mathcal{L}}_\alpha$ is direct, we get that

$$\tilde{\mathcal{L}}_0 = \sum_{\alpha, \beta; \alpha+\beta=0} [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_\beta] = [\tilde{\mathcal{L}}_0, \tilde{\mathcal{L}}_0] + \sum_{\alpha \in R \setminus \{0\}} [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_{-\alpha}].$$

But $\mathcal{L}_0 = \sum_{\alpha \in R \setminus \{0\}} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$, so

$$
\begin{aligned}
[\tilde{\mathcal{L}}_0, \tilde{\mathcal{L}}_0\tilde{]} = [\mathcal{L}_0, \tilde{\mathcal{L}}_0\tilde{]} &= [\sum_{\alpha \in R \setminus \{0\}} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}], \tilde{\mathcal{L}}_0\tilde{]} \\
&= [\sum_{\alpha \in R \setminus \{0\}} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]\tilde{}, \tilde{\mathcal{L}}_0\tilde{]} \\
&= [\sum_{\alpha \in R \setminus \{0\}} [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_{-\alpha}]\tilde{}, \tilde{\mathcal{L}}_0\tilde{]} \\
&\subseteq \sum_{\alpha \in R \setminus \{0\}} ([\tilde{\mathcal{L}}_\alpha, [\tilde{\mathcal{L}}_{-\alpha}, \tilde{\mathcal{L}}_0\tilde{]}\tilde{]} + [\tilde{\mathcal{L}}_{-\alpha}, [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_0\tilde{]}\tilde{]}) \\
&\subseteq \sum_{\alpha \in R \setminus \{0\}} [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_{-\alpha}\tilde{]}.
\end{aligned}
$$

So $\tilde{\mathcal{L}}_0 = \sum_{\alpha \in R \setminus \{0\}} [\tilde{\mathcal{L}}_\alpha, \tilde{\mathcal{L}}_{-\alpha}]$. This shows that $\tilde{\mathcal{L}}$ is an $R-$graded Lie algebra with grading pair $(\mathcal{G}, \mathcal{H})$. Next suppose R is a finite root system. The Lie algebra epimorphism $\pi : \tilde{\mathcal{L}} \longrightarrow \mathcal{L}$ induces a Lie algebra epimorphism $\varphi : \tilde{\mathcal{L}}/Z(\tilde{\mathcal{L}}) \longrightarrow \mathcal{L}/Z(\mathcal{L})$ mapping $\tilde{x} + Z(\tilde{\mathcal{L}})$ to $\pi(\tilde{x}) + Z(\mathcal{L})$ for $\tilde{x} \in \tilde{\mathcal{L}}$. We claim that φ is a Lie algebra isomorphism. Suppose that $\tilde{x} \in \tilde{\mathcal{L}}$ and $\pi(\tilde{x}) \in Z(\mathcal{L})$, then for each $\tilde{y} \in \tilde{\mathcal{L}}$, $\pi([\tilde{x}, \tilde{y}]) = [\pi(\tilde{x}), \pi(\tilde{y})] = 0$ which implies that $[\tilde{x}, \tilde{y}] \in ker(\pi) \subseteq Z(\tilde{\mathcal{L}})$. Now it follows that for each $\tilde{y}, \tilde{z} \in \tilde{\mathcal{L}}$, $[\tilde{x}, [\tilde{y}, \tilde{z}]] = 0$, and so as $\tilde{\mathcal{L}}$ is perfect, we get that $\tilde{x} \in Z(\tilde{\mathcal{L}})$, therefore φ is injective. Now as $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are perfect and φ is an isomorphism, we get that $\mathcal{L}$, $\mathcal{L}/Z(\mathcal{L})$, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}/Z(\tilde{\mathcal{L}})$ have the same universal central extension, say $\mathfrak{A}$. Therefore $\mathcal{L}$ as well as $\tilde{\mathcal{L}}$ are quotient algebras of $\mathfrak{A}$ by subspaces of the center of $\mathfrak{A}$. Now we are done using [1, Thm. 420], [2, Thm. 5.34]. $\qquad\square$

Suppose that $\mathfrak{q} = (\mathfrak{a}, *, \mathcal{C}, f)$ is a coordinate quadruple of type BC and R is an irreducible locally finite root system of type BC_I for an infinite index set I. Take $\mathfrak{b} := \mathfrak{b}(\mathfrak{q})$ to be the algebra corresponding to $\mathfrak{q}$ and suppose $\mathcal{K}$ is a subspace of $\mathrm{HF}(\mathfrak{b})$ satisfying the universal property on $\mathfrak{b}$. Fix a finite subset I_0 of I of cardinality greater than 3 and suppose $\{I_\lambda \mid \lambda \in \Lambda\}$, where Λ is an index set containing zero, is the class of all finite subsets of I containing I_0. For $\lambda \in \Lambda$, suppose R_λ is the finite subsystem of R of type BC_{I_λ}. Next suppose $\mathcal{G} = \sum_{\alpha \in R_{sdiv}} \mathcal{G}_\alpha$ is a locally finite split simple Lie algebra of type C_I with splitting Cartan subalgebra $\mathcal{H}$ and for $\lambda \in \Lambda$, take $\mathcal{G}^\lambda := \sum_{\alpha \in (R_\lambda)^\times_{sdiv}} \mathcal{G}_\alpha \oplus \sum_{\alpha \in (R_\lambda)^\times_{sdiv}} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$. One knows that $\mathcal{G}^\lambda$ is a finite dimensional split simple Lie subalgebra of $\mathcal{G}$ of type $(R_\lambda)_{sdiv}$ and $\mathcal{H}_\lambda := \mathcal{G}^\lambda \cap \mathcal{H}$ is a splitting Cartan subalgebra of $\mathcal{G}^\lambda$. We also recall that $\mathcal{G}$ is the direct union of $\{\mathcal{G}^\lambda \mid \lambda \in \Lambda\}$. Consider the $R-$graded Lie algebra

(1.18) $\qquad \mathcal{L} := \mathcal{L}(\mathfrak{q}, \mathcal{K}) = (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \langle \mathfrak{b}, \mathfrak{b} \rangle$

as in Theorem 1.13 in which $\langle \mathfrak{b}, \mathfrak{b} \rangle := \{\mathfrak{b}, \mathfrak{b}\}/\mathcal{K}$, $\mathcal{S}$ is an irreducible $\mathcal{G}-$module equipped with a weight space decomposition with respect to $\mathcal{H}$ whose set

of weights is $R_{lg} \cup \{0\}$ and $\mathcal{V}$ is an irreducible $\mathcal{G}-$module equipped with a weight space decomposition with respect to $\mathcal{H}$ whose set of weights is R_{sh}. We also recall that $\mathcal{S}$ as a vector space is the direct union of the class $\{\mathcal{S}^\lambda \mid \lambda \in \Lambda\}$ where for $\lambda \in \Lambda$, $\mathcal{S}^\lambda$ is the irreducible finite dimensional $\mathcal{G}^\lambda-$module whose set of weights, with respect to $\mathcal{H}_\lambda$, is $(R_\lambda)_{lg} \cup \{0\}$ and $(\mathcal{S}^\lambda)_\alpha = \mathcal{S}_\alpha$ for $\alpha \in (R^\lambda)_{lg}$, also $\mathcal{V}$ as a vector space is the direct union of the class $\{\mathcal{V}^\lambda \mid \lambda \in \Lambda\}$ where for $\lambda \in \Lambda$, $\mathcal{V}^\lambda$ is the irreducible finite dimensional $\mathcal{G}^\lambda-$module whose set of weights, with respect to $\mathcal{H}_\lambda$, is $(R_\lambda)_{sh}$ and $(\mathcal{V}^\lambda)_\alpha = \mathcal{V}_\alpha$ for $\alpha \in (R^\lambda)_{sh}$ (see (1.12)). We next recall from [8] that for $\lambda \in \Lambda$, there is a subalgebra $\mathcal{D}_\lambda$ of $\mathcal{L}$ with $\mathcal{D}_0 = \langle \mathfrak{b}, \mathfrak{b} \rangle$ such that

$$[\mathcal{G}^\lambda, \mathcal{D}_\lambda] = \{0\},$$

(1.19)
$$\mathcal{D}_\lambda \oplus (\mathcal{S}^\lambda \dot\otimes \mathcal{B}) = \mathcal{D}_0 \oplus (\mathcal{S}^\lambda \dot\otimes \mathcal{B}),$$

$$\begin{aligned}\mathcal{L}^\lambda &:= (\mathcal{G}^\lambda \dot\otimes \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot\otimes \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot\otimes \mathcal{C}) \oplus \mathcal{D}_\lambda \\ &= (\mathcal{G}^\lambda \dot\otimes \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot\otimes \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot\otimes \mathcal{C}) \oplus \mathcal{D}_0\end{aligned}$$

and that $\mathcal{L}^\lambda$ is a Lie algebra graded by R_λ. Suppose that $\pi : (\tilde{\mathcal{L}}, [\cdot, \tilde{\cdot}]) \longrightarrow (\mathcal{L}, [\cdot, \cdot])$ is a central extension of $\mathcal{L}$. As before, we may assume $\tilde{\mathcal{L}} = \mathcal{L} \oplus ker(\pi)$, π is the projection map on $\mathcal{L}$ and there is a 2-cocycle $\tau : \mathcal{L} \times \mathcal{L} \longrightarrow ker(\pi)$ such that

$$[x_1 + z_1, x_2 + z_2\tilde{]} = [x_1, x_2] + \tau(x_1, x_2); \quad x_1, x_2 \in \mathcal{L}, \quad z_1, z_2 \in ker(\pi).$$

We note that

(1.20)
$$\begin{aligned}\tilde{\mathcal{L}}^\lambda &:= (\mathcal{G}^\lambda \dot\otimes \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot\otimes \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot\otimes \mathcal{C}) \oplus \mathcal{D}_\lambda \oplus ker(\pi) \\ &= (\mathcal{G}^\lambda \dot\otimes \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot\otimes \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot\otimes \mathcal{C}) \oplus \mathcal{D}_0 \oplus ker(\pi),\end{aligned}$$

is a central extension of $\mathcal{L}^\lambda$. Now consider the map

$$\cdot : \mathcal{G} \times \tilde{\mathcal{L}} \longrightarrow \tilde{\mathcal{L}}$$
$$x \cdot y \mapsto [x, y\tilde{]}; \quad x \in \mathcal{G}, \ y \in \tilde{\mathcal{L}},$$

which defines a $\mathcal{G}-$module action on $\tilde{\mathcal{L}}$. One can see that π is a $\mathcal{G}-$module homomorphism.

Now for each $\lambda \in \Lambda$, take

(1.21)
$$\mathcal{E}_\lambda := \mathcal{G}^\lambda \oplus \tau(\mathcal{G}, \mathcal{G}).$$

One can see that $\mathcal{E}_\lambda$ is a $\mathcal{G}^\lambda-$module via the action "$\cdot$" restricted to $\mathcal{G}^\lambda \times \mathcal{E}_\lambda$. Now as each finite subset $\{x_1 + r_1, \ldots, x_n + r_n \mid x_i \in \mathcal{G}^\lambda, \ r_i \in \tau(\mathcal{G}, \mathcal{G}); \ 1 \leq i \leq n\}$ of $\mathcal{E}_\lambda$ is contained in $\mathcal{G}^\lambda \oplus (\tau(\mathcal{G}^\lambda, \mathcal{G}^\lambda) + \text{span}\{r_1, \ldots, r_n\})$ which is a finite dimensional $\mathcal{G}^\lambda-$submodule of the $\mathcal{G}^\lambda-$module $\mathcal{E}_\lambda$, we get that $\mathcal{E}_\lambda$ is a locally finite $\mathcal{G}^\lambda-$module. Therefore it is completely reducible as $\mathcal{G}^\lambda$ is a finite dimensional split simple Lie algebra. Now as $\tau(\mathcal{G}, \mathcal{G})$ is a $\mathcal{G}^0-$submodule of $\mathcal{E}_0$, there is a $\mathcal{G}^0-$submodule $\dot{\mathcal{G}}^0$ of $\mathcal{E}_0$ such that $\mathcal{E}_0 =$

$\dot{\mathcal{G}}^0 \oplus \tau(\mathcal{G}, \mathcal{G})$. Next we note that $\mathcal{E}_0 \subseteq \mathcal{E}_\lambda$ and for $\lambda \in \Lambda$, define

(1.22) $\qquad \dot{\mathcal{G}}^\lambda :=$ the $\mathcal{G}^\lambda-$submodule of $\mathcal{E}_\lambda$ generated by $\dot{\mathcal{G}}^0$.

Lemma 1.23. *(i) Set $\mathcal{E} := \mathcal{G} \oplus \tau(\mathcal{G}, \mathcal{G})$, then $\mathcal{E}$ is both a Lie subalgebra and a $\mathcal{G}-$submodule of $\tilde{\mathcal{L}}$. Also the restriction of π to $\mathcal{E}$ is both a Lie algebra homomorphism and a $\mathcal{G}-$module homomorphism.*

(ii) $\dot{\mathcal{G}}^0$ is a Lie subalgebra of $\tilde{\mathcal{L}}$ and the restriction of π to $\dot{\mathcal{G}}^0$ is both a Lie algebra isomorphism and a $\mathcal{G}^0-$module isomorphism from $\dot{\mathcal{G}}^0$ onto $\mathcal{G}^0$. In particular, $\dot{\mathcal{G}}^0$ is a Lie subalgebra of $\tilde{\mathcal{L}}$ isomorphic to $\mathcal{G}^0$ as well as an irreducible $\mathcal{G}^0-$submodule of $\mathcal{E}_0$.

Proof. (i) It is trivial.

(ii) Suppose that $a, b \in \dot{\mathcal{G}}^0$, then since $\mathcal{G}^0 \oplus \tau(\mathcal{G}, \mathcal{G}) = \dot{\mathcal{G}}^0 \oplus \tau(\mathcal{G}, \mathcal{G})$, there are unique $x, y \in \mathcal{G}^0$, and $r, s \in \tau(\mathcal{G}, \mathcal{G})$ such that $a = x + r$ and $b = y + s$. Now as $\dot{\mathcal{G}}^0$ is a $\mathcal{G}^0-$submodule of $\mathcal{E}_0$, we get that $[a, \tilde{b}] = [x, \tilde{b}] \in \dot{\mathcal{G}}^0$. So $\dot{\mathcal{G}}^0$ is a Lie subalgebra of $\tilde{\mathcal{L}}$. Next we show that $\pi_0 := \pi|_{\dot{\mathcal{G}}^0}$ is one to one. Suppose that $a, b \in \dot{\mathcal{G}}^0$ and $\pi_0(a) = \pi_0(b)$. Since $\dot{\mathcal{G}}^0 \oplus \tau(\mathcal{G}, \mathcal{G}) = \mathcal{G}^0 \oplus \tau(\mathcal{G}, \mathcal{G})$, we get that $\pi_0(a) = \pi_0(b) \in \mathcal{G}^0$ and that there are unique $r, s \in \tau(\mathcal{G}, \mathcal{G})$ such that $a = \pi_0(a) + r$ and $b = \pi_0(b) + s$. Now as $\pi_0(a) = \pi_0(b)$ and $\dot{\mathcal{G}}^0 \cap \tau(\mathcal{G}, \mathcal{G}) = \{0\}$, we get that $a - b = r - s = 0$. Now we are done using the fact that $\dot{\mathcal{G}}^0 \oplus \tau(\mathcal{G}, \mathcal{G}) = \mathcal{G}^0 \oplus \tau(\mathcal{G}, \mathcal{G})$. $\qquad \square$

Lemma 1.24. *Recall (1.22), we have for $\lambda \in \Lambda$ that $\dot{\mathcal{G}}^\lambda$ is a Lie subalgebra of $\tilde{\mathcal{L}}$ and the restriction of π to $\dot{\mathcal{G}}^\lambda$ is both a Lie algebra isomorphism and a $\mathcal{G}^\lambda-$module isomorphism from $\dot{\mathcal{G}}^\lambda$ to $\mathcal{G}^\lambda$. In particular, $\dot{\mathcal{G}}^\lambda$ is a Lie subalgebra of $\tilde{\mathcal{L}}$ isomorphic to $\mathcal{G}^\lambda$ and it is an irreducible $\mathcal{G}^\lambda-$submodule of $\tilde{\mathcal{L}}$ isomorphic to $\mathcal{G}^\lambda$. Moreover $\mathcal{G}^\lambda \oplus \tau(\mathcal{G}, \mathcal{G}) = \dot{\mathcal{G}}^\lambda \oplus \tau(\mathcal{G}, \mathcal{G})$.*

Proof. We know that $\mathcal{E}_\lambda = \mathcal{G}^\lambda \oplus \tau(\mathcal{G}, \mathcal{G})$ is a locally finite $\mathcal{G}^\lambda-$submodule of $\tilde{\mathcal{L}}$ under the action "$\cdot$" restricted to $\mathcal{G}^\lambda \times \mathcal{E}_\lambda$ and that $\tau(\mathcal{G}, \mathcal{G})$ is a submodule of $\mathcal{G}^\lambda \oplus \tau(\mathcal{G}, \mathcal{G})$. Therefore there is a $\mathcal{G}^\lambda-$submodule $\mathcal{P}$ of $\mathcal{G}^\lambda \oplus \tau(\mathcal{G}, \mathcal{G})$ such that $\mathcal{E}_\lambda = \mathcal{G}^\lambda \oplus \tau(\mathcal{G}, \mathcal{G}) = \mathcal{P} \oplus \tau(\mathcal{G}, \mathcal{G})$. Then setting $\theta := \pi|_{\mathcal{P}}$, we get using the same argument as in Lemma 1.23, that $\theta : \mathcal{P} \longrightarrow \mathcal{G}^\lambda$ is a Lie algebra isomorphism and also a $\mathcal{G}^\lambda-$module isomorphism. Thus as $\mathcal{G}^\lambda$ is equipped with a weight space decomposition $\mathcal{G}^\lambda = (\mathcal{G}^\lambda)_0 \oplus \sum_{\alpha \in (R_\lambda)^\times_{sdiv}} (\mathcal{G}^\lambda)_\alpha$ with respect to $\mathcal{H}_\lambda$, we have the weight space decomposition $\mathcal{P} = \mathcal{P}_0 \oplus \sum_{\alpha \in (R_\lambda)^\times_{sdiv}} \mathcal{P}_\alpha$ for $\mathcal{P}$ with respect to $\mathcal{H}_\lambda$, where $\mathcal{P}_\alpha := \theta^{-1}((\mathcal{G}^\lambda)_\alpha)$ for $\alpha \in (R_\lambda)_{sdiv}$. This in turn implies that $(\mathcal{P}_0 \oplus \tau(\mathcal{G}, \mathcal{G})) \oplus \sum_{\alpha \in (R_\lambda)^\times_{sdiv}} \mathcal{P}_\alpha$ is a weight space decomposition for $\mathcal{E}_\lambda$ with respect to $\mathcal{H}_\lambda$. We next note $\mathcal{G}^0$ has a weight space decomposition $\mathcal{G}^0 = \sum_{\alpha \in (R_0)_{sdiv}} (\mathcal{G}^0)_\alpha$ with respect to $\mathcal{H}_0$ where $(\mathcal{G}^0)_0 = \sum_{\alpha \in (R_0)^\times_{sdiv}} [(\mathcal{G}^\lambda)_\alpha, (\mathcal{G}^\lambda)_{-\alpha}]$ and for $\alpha \in (R_0)^\times_{sdiv}$, $(\mathcal{G}^0)_\alpha = (\mathcal{G}^\lambda)_\alpha$. Setting $\mathcal{Q} := \theta^{-1}(\mathcal{G}^0)$ and $\mathcal{Q}_\alpha := \theta^{-1}((\mathcal{G}^0)_\alpha) = \mathcal{P}_\alpha$ for $\alpha \in (R_0)_{sdiv} \setminus \{0\}$, one gets that $\mathcal{Q}$ is a $\mathcal{G}^0-$submodule of $\mathcal{P}$ isomorphic to $\mathcal{G}^0$ and equipped with the weight space decomposition $\mathcal{Q} = \sum_{\alpha \in (R_0)^\times_{sdiv}} \mathcal{Q}_\alpha \oplus \sum_{\alpha \in (R_0)^\times_{sdiv}} [\mathcal{Q}_\alpha, \mathcal{Q}_{-\alpha}]$

with respect to $\mathcal{H}_0$. Also $\sum_{\alpha \in (R_0)^{\times}_{sdiv}} \mathcal{Q}_\alpha \oplus (\sum_{\alpha \in (R_0)^{\times}_{sdiv}} [\mathcal{Q}_\alpha, \mathcal{Q}_{-\alpha}] \oplus \tau(\mathcal{G}, \mathcal{G}))$ and $\mathcal{E}_0 = \mathcal{Q} \oplus \tau(\mathcal{G}, \mathcal{G})$ is a weight space decomposition of $\mathcal{E}_0$ with respect to $\mathcal{H}_0$. Now $\dot{\mathcal{G}}^0$ is a nontrivial finite dimensional irreducible $\mathcal{G}^0$−submodule of $\mathcal{E}_0$ isomorphic to $\mathcal{G}^0$ and so by [Y,Theorem], $\dot{\mathcal{G}}^\lambda$ is a $\mathcal{G}^\lambda$−submodule of $\mathcal{E}_\lambda$ isomorphic to $\mathcal{G}^\lambda$. On the other hand we know that $\theta : \mathcal{E}_\lambda \longrightarrow \mathcal{G}^\lambda$ is a $\mathcal{G}^\lambda$−module homomorphism. Now as $\mathcal{G}^\lambda$ and $\dot{\mathcal{G}}^\lambda$ are irreducible $\mathcal{G}^\lambda$−modules and $\theta(\dot{\mathcal{G}}^0) = \pi(\dot{\mathcal{G}}^0) \neq 0$, one gets that the restriction of π to $\dot{\mathcal{G}}^\lambda$ is a $\mathcal{G}^\lambda$−module isomorphism from $\dot{\mathcal{G}}^\lambda$ onto $\mathcal{G}^\lambda$ which in turn implies that $\dot{\mathcal{G}}^\lambda \oplus \tau(\mathcal{G}, \mathcal{G}) = \mathcal{G}^\lambda \oplus \tau(\mathcal{G}, \mathcal{G})$ and that $\pi \mid_{\dot{\mathcal{G}}_\lambda} : \dot{\mathcal{G}}^\lambda \longrightarrow \mathcal{G}^\lambda$ is also a Lie algebra isomorphism. $\square$

Corollary 1.25. *For $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$, we have $\dot{\mathcal{G}}^\lambda \subseteq \dot{\mathcal{G}}^\mu$, in particular $\cup_{\lambda \in \Lambda} \dot{\mathcal{G}}^\lambda$ is a subalgebra of $\tilde{\mathcal{L}}$ and also a $\mathcal{G}$−submodule of $\tilde{\tilde{\mathcal{L}}}$. Also setting $\dot{\mathcal{G}}$ to be the direct union of $\{\dot{\mathcal{G}}^\lambda \mid \lambda \in \Lambda\}$, $\pi \mid_{\dot{\mathcal{G}}}$ is both a Lie algebra isomorphism and a $\mathcal{G}$−module isomorphism from $\dot{\mathcal{G}}$ to $\mathcal{G}$. Moreover, we have $\mathcal{G} \oplus \tau(\mathcal{G}, \mathcal{G}) = \dot{\mathcal{G}} \oplus \tau(\mathcal{G}, \mathcal{G})$.*

Proof. Since $\dot{\mathcal{G}}^0 \subseteq \dot{\mathcal{G}}^\lambda \cap \dot{\mathcal{G}}^\mu$ and $\mathcal{G}^\lambda \subseteq \mathcal{G}^\mu$, we get that $\dot{\mathcal{G}}^\lambda \subseteq \dot{\mathcal{G}}^\mu$. Now using Lemmas 1.23 and 1.24, we are done. $\square$

Recall (1.18) and suppose $\mathcal{I}$ is an index set containing zero and fix a basis $\{a_i \mid i \in \mathcal{I}\}$ with $a_0 = 1$ for $\mathcal{A}$, also fix a basis $\{b_j \mid j \in \mathcal{J}\}$ for $\mathcal{B}$ and a basis $\{c_t \mid t \in \mathcal{T}\}$ for $\mathcal{C}$. For $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $t \in \mathcal{T}$, set

(1.26) $$\mathcal{G}_i := \mathcal{G} \otimes a_i, \quad \mathcal{S}_j := \mathcal{S} \otimes b_j, \quad \mathcal{V}_t := \mathcal{V} \otimes c_t$$

which are $\mathcal{G}$−submodules of $\mathcal{L}$; also for $\lambda \in \Lambda$, set

(1.27) $$\mathcal{G}_i^\lambda := \mathcal{G}^\lambda \otimes a_i, \quad \mathcal{S}_j^\lambda := \mathcal{S}^\lambda \otimes b_j, \quad \mathcal{V}_t^\lambda := \mathcal{V}^\lambda \otimes c_t.$$

Now suppose $\mathcal{M}$ is one of the $\mathcal{G}$−submodules of $\mathcal{L}$ in the class $\{\mathcal{G}_i, \mathcal{S}_j, \mathcal{V}_t \mid i \in \mathcal{I} \setminus \{0\}, j \in \mathcal{J}, t \in \mathcal{T}\}$ and consider $\tilde{\mathcal{M}} := \mathcal{M} \oplus \tau(\mathcal{G}, \mathcal{M})$. Then $\tilde{\mathcal{M}}$ is a $\mathcal{G}$−submodule of $\tilde{\mathcal{L}}$. We know that the vector space $\mathcal{M}$ is the direct union of a class $\{\mathcal{M}^\lambda \mid \lambda \in \Lambda\}$ in which each $\mathcal{M}^\lambda$ is a finite dimensional irreducible $\mathcal{G}^\lambda$−submodule of $\mathcal{L}^\lambda$ equipped with the weight spaced decomposition $\mathcal{M}^\lambda = \oplus_{\gamma \in \Gamma_\lambda} (\mathcal{M}^\lambda)_\gamma$ where
(1.28)

- $\Gamma_\lambda \subseteq R_\lambda$,
- For $* \in \{sh, lg, ex\}$, $\Gamma_\lambda \setminus \{0\} = (R_\lambda)_*$ if and only if $\Gamma_0 \setminus \{0\} = (R_0)_*$,
- $\Gamma_\lambda \subseteq \Gamma_\mu; \quad \lambda \prec \mu$,
- $(\mathcal{M}^\lambda)_\gamma = (\mathcal{M}^\mu)_\gamma; \lambda \prec \mu, \gamma \in \Gamma_\lambda \setminus \{0\}$.

For $\lambda \in \Lambda$, set $\tilde{\mathcal{M}}^\lambda := \mathcal{M}^\lambda \oplus \tau(\mathcal{G}, \mathcal{M})$, then $\tilde{\mathcal{M}}^\lambda$ is a $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}$ under the action "$\cdot$" restricted to $\mathcal{G}^\lambda \times \tilde{\mathcal{L}}$. Now if $\{m_i + r_i \mid 1 \leq i \leq n, m_i \in \mathcal{M}^\lambda, r_i \in \tau(\mathcal{M}, \mathcal{G})\}$ is a finite subset of $\tilde{\mathcal{M}}^\lambda$, we see that $\{m_i + r_i \mid 1 \leq i \leq n\} \subseteq \mathcal{M}^\lambda + \tau(\mathcal{G}^\lambda, \mathcal{M}^\lambda) + \text{span}_{\mathbb{F}}\{r_1, \ldots, r_n\}$ which is a finite dimensional $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}$. This means that $\tilde{\mathcal{M}}^\lambda$ is a locally finite $\mathcal{G}^\lambda$−module and so it is completely reducible as $\mathcal{G}^\lambda$ is finite dimensional simple Lie algebra.

Next we note that $\tau(\mathcal{G}, \mathcal{M})$ is a $\mathcal{G}^0-$submodule of the locally finite $\mathcal{G}^0-$module $\tilde{\mathcal{M}}^0$, so there is a $\mathcal{G}^0-$submodule $\dot{\mathcal{M}}^0$ of $\tilde{\mathcal{M}}^0$ such that $\tilde{\mathcal{M}}^0 = \dot{\mathcal{M}}^0 \oplus \tau(\mathcal{G}, \mathcal{M})$.
Set

(1.29) $\dot{\mathcal{M}}^\lambda := \mathcal{G}^\lambda-$submodule of $\tilde{\mathcal{M}}^\lambda$ generated by $\dot{\mathcal{M}}^0$; $\lambda \in \Lambda$.

Lemma 1.30. *(i) For $\lambda \in \Lambda$, the restriction of π to $\dot{\mathcal{M}}^\lambda$ is a $\mathcal{G}^\lambda-$module isomorphism from $\dot{\mathcal{M}}^\lambda$ onto $\mathcal{M}^\lambda$ and $\tilde{\mathcal{M}}^\lambda = \dot{\mathcal{M}}^\lambda \oplus \tau(\mathcal{G}, \mathcal{M})$.*

(ii) For $\lambda \prec \mu$, we have $\dot{\mathcal{M}}^\lambda \subseteq \dot{\mathcal{M}}^\mu$, in particular $\dot{\mathcal{M}}$, the direct union of $\{\dot{\mathcal{M}}^\lambda \mid \lambda \in \Lambda\}$, is a $\mathcal{G}-$submodule of $\tilde{\mathcal{L}}$. Also the restriction of π to $\dot{\mathcal{M}}$ is a $\mathcal{G}-$module isomorphism from $\dot{\mathcal{M}}$ onto $\mathcal{M}$ and $\tilde{\mathcal{M}} = \dot{\mathcal{M}} \oplus \tau(\mathcal{G}, \mathcal{M}) = \mathcal{M} \oplus \tau(\mathcal{G}, \mathcal{M})$.

(iii) If $x \in \dot{\mathcal{M}}$ and $\pi(x) \in \mathcal{M}^\lambda$ for some $\lambda \in \Lambda$, then $x \in \dot{\mathcal{M}}^\lambda$.

Proof. (i) We first note that since π is a $\mathcal{G}-$module homomorphism, the restriction of π to $\tilde{\mathcal{M}}$ is a $\mathcal{G}-$module homomorphism. Now as $\mathcal{M}^0 \oplus \tau(\mathcal{G}, \mathcal{M}) = \dot{\mathcal{M}}^0 \oplus \tau(\mathcal{G}, \mathcal{M})$ and that $\mathcal{M}^0$ is an irreducible $\mathcal{G}^0-$module, it is immediate that the restriction of π to $\dot{\mathcal{M}}^0$ is a $\mathcal{G}^0-$module isomorphism from $\dot{\mathcal{M}}^0$ onto $\mathcal{M}^0$. Now suppose that $0 \prec \lambda$, since $\tilde{\mathcal{M}}^\lambda$ is a completely reducible $\mathcal{G}^\lambda-$module and $\tau(\mathcal{G}, \mathcal{M})$ is a $\mathcal{G}^\lambda-$submodule of $\tilde{\mathcal{M}}^\lambda$, one finds a $\mathcal{G}^\lambda-$submodule of $\mathcal{N}$ of $\tilde{\mathcal{M}}^\lambda$ such that $\tilde{\mathcal{M}}^\lambda = \mathcal{N} \oplus \tau(\mathcal{G}, \mathcal{M})$. Therefore $\theta := \pi|_{\mathcal{N}} : \mathcal{N} \longrightarrow \mathcal{M}^\lambda$ is a $\mathcal{G}^\lambda-$module isomorphism. We know that $\mathcal{M}^\lambda$ has a weight space decomposition $\mathcal{M}^\lambda = \oplus_{\alpha \in \Gamma_\lambda}(\mathcal{M}^\lambda)_\alpha$ with respect to $\mathcal{H}_\lambda$ and that $\mathcal{M}^0$ has a weight space decomposition $\mathcal{M}^0 = \oplus_{\alpha \in \Gamma_0}(\mathcal{M}^0)_\alpha$ with respect to $\mathcal{H}_0$ such that
(1.31)

$\quad \Gamma_0 \subseteq R_0, \quad \Gamma_\lambda \subseteq R_\lambda,$
$\quad$ For $* \in \{sh, lg, ex\}$, $\Gamma_\lambda \setminus \{0\} = (R_\lambda)_*$ if and only if $\Gamma_0 \setminus \{0\} = (R_0)_*,$
$\quad (\mathcal{M}^0)_\alpha = (\mathcal{M}^\lambda)_\alpha$ for $\alpha \in \Gamma_0 \setminus \{0\},$

(see (1.28)). Now since $\mathcal{M}^0$ is a $\mathcal{G}^0-$submodule of $\mathcal{M}^\lambda$ and θ is a $\mathcal{G}^\lambda-$module isomorphism, $\mathcal{N}^0 := \theta^{-1}(\mathcal{M}^0) \subseteq \mathcal{M}^0 \oplus \tau(\mathcal{G}, \mathcal{M}) = \tilde{\mathcal{M}}^0$ is a $\mathcal{G}^0-$submodule of $\tilde{\mathcal{M}}^0$. Also $\mathcal{N}$ is a $\mathcal{G}^\lambda-$module equipped with the weight space decomposition $\mathcal{N} = \oplus_{\alpha \in \Gamma_\lambda} \mathcal{N}_\alpha$ with respect to $\mathcal{H}_\lambda$, where for $\alpha \in \Gamma_\lambda$, $\mathcal{N}_\alpha := \theta^{-1}((\mathcal{M}^\lambda)_\alpha)$, and that $\mathcal{N}^0$ has a weight space decomposition $\mathcal{N}^0 = \oplus_{\alpha \in \Gamma_0}(\mathcal{N}^0)_\alpha$, with respect to $\mathcal{H}_0$, where for $\alpha \in \Gamma_0$, $(\mathcal{N}^0)_\alpha := \theta^{-1}((\mathcal{M}^0)_\alpha)$. Therefore $\tilde{\mathcal{M}}^\lambda$ has a weight space decomposition $\tilde{\mathcal{M}}^\lambda = \oplus_{\alpha \in \Gamma_\lambda \cup \{0\}}(\tilde{\mathcal{M}}^\lambda)_\alpha$, with respect to $\mathcal{H}_\lambda$ where

(1.32) $(\tilde{\mathcal{M}}^\lambda)_\alpha = \begin{cases} (\mathcal{M}^\lambda)_\alpha & \text{if } \alpha \in \Gamma_\lambda \setminus \{0\}, \\ (\mathcal{M}^\lambda)_0 + \tau(\mathcal{G}, \mathcal{M}) & \text{if } \alpha = 0 \text{ and } 0 \in \Gamma_\lambda, \\ \tau(\mathcal{G}, \mathcal{M}) & \text{if } \alpha = 0 \text{ and } 0 \notin \Gamma_\lambda. \end{cases}$

Also $\tilde{\mathcal{M}}^0 = \mathcal{N}^0 \oplus \tau(\mathcal{G}, \mathcal{M})$ and $\tilde{\mathcal{M}}^0$ is equipped with the weight space decomposition $\tilde{\mathcal{M}}^0 = \oplus_{\alpha \in \Gamma_0 \cup \{0\}} (\tilde{\mathcal{M}}^0)_\alpha$ with respect to $\mathcal{H}_0$, where

$$(1.33) \qquad (\tilde{\mathcal{M}}^0)_\alpha = \begin{cases} (\mathcal{M}^0)_\alpha & \text{if } \alpha \in \Gamma_0 \setminus \{0\}, \\ (\mathcal{M}^0)_0 + \tau(\mathcal{G}, \mathcal{M}) & \text{if } \alpha = 0 \text{ and } 0 \in \Gamma_0, \\ \tau(\mathcal{G}, \mathcal{M}) & \text{if } \alpha = 0 \text{ and } 0 \notin \Gamma_0. \end{cases}$$

Now (1.31)-(1.33) together with [Y, Theorem] imply that the $\mathcal{G}^\lambda$−submodule $\dot{\mathcal{M}}^\lambda$ of $\tilde{\mathcal{M}}^\lambda$ generated by $\dot{\mathcal{M}}^0$ is a $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{M}}^\lambda$ isomorphic to $\mathcal{M}^\lambda$. This together with the facts that $\pi(\dot{\mathcal{M}}^\lambda) \subseteq \mathcal{M}^\lambda$, $\pi(\dot{\mathcal{M}}^0) \neq \{0\}$, and $\dot{\mathcal{M}}^\lambda$ as well as $\mathcal{M}^\lambda$ are irreducible $\mathcal{G}^\lambda$−modules, implies that the restriction of π to $\dot{\mathcal{M}}^\lambda$ is a $\mathcal{G}^\lambda$−module isomorphism from $\dot{\mathcal{M}}^\lambda$ to $\mathcal{M}^\lambda$. In particular, we get that $\tilde{\mathcal{M}}^\lambda = \mathcal{M}^\lambda \oplus \tau(\mathcal{G}, \mathcal{M}) = \dot{\mathcal{M}}^\lambda \oplus \tau(\mathcal{G}, \mathcal{M})$.

(ii) This is easy to see using Part (i).

(iii) Take $\mu \in \Lambda$ to be such that $x \in \dot{\mathcal{M}}^\mu$. We know that there is $\nu \in \Lambda$ with $\lambda \prec \nu$ and $\mu \prec \nu$. Since the restriction of π to $\dot{\mathcal{M}}^\lambda$ is a $\mathcal{G}^\lambda$−module isomorphism from $\dot{\mathcal{M}}^\lambda$ onto $\mathcal{M}^\lambda$, one finds $y \in \dot{\mathcal{M}}^\lambda$ such that $\pi(x) = \pi(y)$. So $x, y \in \dot{\mathcal{M}}^\nu$ and $\pi(x) = \pi(y)$. But the restriction of π to $\dot{\mathcal{M}}^\nu$ is a $\mathcal{G}^\nu$−module isomorphism from $\dot{\mathcal{M}}^\nu$ onto $\mathcal{M}^\nu$, therefore $x = y \in \dot{\mathcal{M}}^\lambda$. $\qquad\square$

Consider (1.26) and (1.27) and identify $\mathcal{G} \otimes 1$ with $\mathcal{G}$. Using Lemmas 1.30, 1.23 and 1.24, if $i \in \mathcal{I} \setminus$, $j \in \mathcal{J}$ and $t \in \mathcal{T}$, for $\lambda \in \Lambda$, one finds irreducible $\mathcal{G}^\lambda$−submodules $\dot{\mathcal{G}}_i^\lambda$, $\dot{\mathcal{S}}_j^\lambda$ and $\dot{\mathcal{V}}_t^\lambda$ of $\tilde{\mathcal{L}}$ such that $\dot{\mathcal{G}}_i^\lambda$ is isomorphic to $\mathcal{G}_i^\lambda$, $\dot{\mathcal{S}}_j^\lambda$ is isomorphic to $\mathcal{S}_j^\lambda$ and $\dot{\mathcal{V}}_t^\lambda$ is isomorphic to $\mathcal{V}_t^\lambda$. Moreover

(1.34)
- $\dot{\mathcal{G}}_i^\lambda$ is the $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}$ generated by $\dot{\mathcal{G}}_i^0$,
- $\dot{\mathcal{S}}_j^\lambda$ is the $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}$ generated by $\dot{\mathcal{S}}_j^0$,
- $\dot{\mathcal{V}}_t^\lambda$ is the $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}$ generated by $\dot{\mathcal{V}}_t^0$.

Also setting $\dot{\mathcal{G}}_i := \varinjlim_{\lambda \in \Lambda} \dot{\mathcal{G}}_i^\lambda$, $\dot{\mathcal{S}}_j := \varinjlim_{\lambda \in \Lambda} \dot{\mathcal{S}}_j^\lambda$ and $\dot{\mathcal{V}}_t := \varinjlim_{\lambda \in \Lambda} \dot{\mathcal{V}}_t^\lambda$, $\dot{\mathcal{G}}_i$ is isomorphic to $\mathcal{G}_i$, $\dot{\mathcal{S}}_j$ is isomorphic to $\mathcal{S}_j$ and $\dot{\mathcal{V}}_t$ is isomorphic to $\mathcal{V}_t$. Also we have

(1.35)
- $\mathcal{G}_i \oplus \tau(\mathcal{G}, \mathcal{G}_i) = \dot{\mathcal{G}}_i \oplus \tau(\mathcal{G}, \mathcal{G}_i)$, $\mathcal{G}_i^\lambda \oplus \tau(\mathcal{G}, \mathcal{G}_i) = \dot{\mathcal{G}}_i^\lambda \oplus \tau(\mathcal{G}, \mathcal{G}_i)$,
- $\mathcal{S}_j \oplus \tau(\mathcal{G}, \mathcal{S}_j) = \dot{\mathcal{S}}_j \oplus \tau(\mathcal{G}, \mathcal{S}_j)$, $\mathcal{S}_j^\lambda \oplus \tau(\mathcal{G}, \mathcal{S}_j) = \dot{\mathcal{S}}_j^\lambda \oplus \tau(\mathcal{G}, \mathcal{S}_j)$,
- $\mathcal{V}_t \oplus \tau(\mathcal{G}, \mathcal{V}_t) = \dot{\mathcal{V}}_t \oplus \tau(\mathcal{G}, \mathcal{V}_t)$, $\mathcal{V}_t^\lambda \oplus \tau(\mathcal{G}, \mathcal{V}_t) = \dot{\mathcal{V}}_t^\lambda \oplus \tau(\mathcal{G}, \mathcal{V}_t)$.

Lemma 1.36. *Consider (1.19), there is a subspace $\dot{\mathcal{D}}$ of $[\mathcal{L}^0, \mathcal{L}^0]^\sim \cap (\mathcal{D}_0 \oplus ker(\pi))$ such that $\pi(\dot{\mathcal{D}}) \subseteq \mathcal{D}_0$, $\pi|_{\dot{\mathcal{D}}} : \dot{\mathcal{D}} \longrightarrow \mathcal{D}_0$ is a linear isomorphism, $[\mathcal{G}^0, \dot{\mathcal{D}}]^\sim = \{0\}$ and for $\lambda \in \Lambda$, $[\mathcal{G}^\lambda, \dot{\mathcal{D}}]^\sim \subseteq \sum_{j \in \mathcal{J}} \dot{\mathcal{S}}_j^\lambda$.*

Proof. We note that $\mathcal{D}_0 \subseteq \mathcal{L}^0 = [\mathcal{L}^0 \mathcal{L}^0]$, so for $d \in \mathcal{D}_0$, there are $n \in \mathbb{N} \setminus \{0\}$, $x_i, y_i \in \mathcal{L}^0$ such that $d = \sum_{i=1}^n [x_i, y_i]$. So we have $\sum_{i=1}^n [x_i, y_i]^\sim = d + \sum_{i=1}^n \tau(x_i, y_i) \in [\mathcal{L}^0, \mathcal{L}^0]^\sim \cap (\mathcal{D}_0 + ker(\pi))$. Also $\pi(\sum_{i=1}^n [x_i, y_i]^\sim) = d$. Therefore there is a subspace $\dot{\mathcal{D}}$ of $[\mathcal{L}^0, \mathcal{L}^0]^\sim \cap (\mathcal{D}_0 \oplus ker(\pi))$ such that $\pi|_{\dot{\mathcal{D}}} : \dot{\mathcal{D}} \longrightarrow \mathcal{D}_0$ is a linear isomorphism. Now using Lemma 1.15 together with

the fact that $\pi \mid_{\mathcal{L}^0 \oplus ker(\pi)} : \mathcal{L}^0 \oplus ker(\pi) \longrightarrow \mathcal{L}^0$ is a central extension of $\mathcal{L}^0$, we get that $[\mathcal{G}^0, \dot{\tilde{\mathcal{D}}}] \subseteq [\mathcal{G}^0, \tilde{\mathcal{D}_0}] = \{0\}$. Next suppose $\lambda \in \Lambda$, $x \in \mathcal{G}^\lambda$ and $\dot{d} \in \dot{\mathcal{D}} \subseteq \mathcal{D}_0 \oplus ker(\pi) \subseteq \mathcal{D}_\lambda + \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda + ker(\pi)$ (see (1.19)). So as $\mathcal{L}^\lambda \oplus ker(\pi)$ is a central extension for $\mathcal{L}^\lambda$, using Lemma 1.15 together with (1.35), we have

$$[x, \dot{\tilde{d}}] \in [x, \mathcal{D}_\lambda + \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda + ker(\pi)\tilde{]} \subseteq [x, \mathcal{D}_\lambda + \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda\tilde{]} \subseteq [x, \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda\tilde{]}$$
$$\subseteq [x, \sum_{j \in \mathcal{J}} \dot{\mathcal{S}}_j^\lambda\tilde{]}$$
$$\subseteq \sum_{j \in \mathcal{J}} \dot{\mathcal{S}}_j^\lambda.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 1.37. $\sum_{i \in I} \dot{\mathcal{G}}_i + \sum_{j \in J} \dot{\mathcal{S}}_j + \sum_{t \in \mathcal{T}} \dot{\mathcal{V}}_t + \dot{\mathcal{D}}$ is a direct sum.

Proof. Suppose that $\sum_i \dot{x}_i \in \sum_{i \in \mathcal{I}} \dot{\mathcal{G}}_i, \sum_j \dot{y}_j \in \sum_{j \in J} \dot{\mathcal{S}}_j, \sum_t \dot{z}_t \in \sum_{t \in \mathcal{T}} \dot{\mathcal{V}}_t, \dot{d} \in \dot{\mathcal{D}}$ and $\sum_i \dot{x}_i + \sum_j \dot{y}_j + \sum_t \dot{z}_t + \dot{d} = 0$, then we have

$$0 = \pi(\sum_i \dot{x}_i + \sum_j \dot{y}_j + \sum_t \dot{z}_t + \dot{d}) = \sum_i \pi(\dot{x}_i) + \sum_j \pi(\dot{y}_j) + \sum_t \pi(\dot{z}_t) + \pi(\dot{d}).$$

Now as $\pi(\dot{\mathcal{G}}_i) = \mathcal{G}_i$, $\pi(\dot{\mathcal{S}}_j) = \mathcal{S}_j$, $\pi(\dot{\mathcal{V}}_t) = \mathcal{V}_t$ and $\sum_{i \in \mathcal{I}} \mathcal{G}_i + \sum_{j \in J} \mathcal{S}_j \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t + \mathcal{D}$ is direct, we get that $\pi(\dot{x}_i) = 0, \pi(\dot{y}_j) = 0, \pi(\dot{z}_t) = 0$ and $\pi(\dot{d}) = 0$. But for $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $t \in \mathcal{T}$, $\pi \mid_{\dot{\mathcal{G}}_i}, \pi \mid_{\dot{\mathcal{S}}_j}, \pi \mid_{\dot{\mathcal{V}}_t}$ and $\pi \mid_{\dot{\mathcal{D}}}$ are isomorphism, so $\dot{x}_i = 0, \dot{y}_j = 0, \dot{z}_t = 0, \dot{d} = 0$ ($i \in \mathcal{I}$, $j \in \mathcal{J}$ and $t \in \mathcal{T}$). $\quad\square$

Now set

(1.38) $$\dot{\mathcal{L}} := \bigoplus_{i \in \mathcal{I}} \dot{\mathcal{G}}_i \oplus \bigoplus_{j \in \mathcal{J}} \dot{\mathcal{S}}_j \oplus \bigoplus_{t \in \mathcal{T}} \dot{\mathcal{V}}_t \oplus \dot{\mathcal{D}}.$$

Using the same argument as in Lemma 1.37, $\dot{\mathcal{L}} \cap ker(\pi) = \{0\}$ and so $\tilde{\mathcal{L}} = \dot{\mathcal{L}} \oplus ker(\pi)$.

Now set

(1.39) $$\pi_1 : \tilde{\mathcal{L}} \longrightarrow \dot{\mathcal{L}} \quad \text{and} \quad \pi_2 : \tilde{\mathcal{L}} \longrightarrow ker(\pi)$$

to be the projective maps on $\dot{\mathcal{L}}$ and $ker(\pi)$ respectively and for $x, y \in \dot{\mathcal{L}}$, define

(1.40) $$[x, y\dot{]} := \pi_1([x, y\tilde{]}) \quad \text{and} \quad \dot{\tau}(x, y) = \pi_2([x, y\tilde{]}).$$

Then we get that $(\dot{\mathcal{L}}, [\cdot, \cdot\dot{]})$ is a Lie algebra and by Lemma 1.24, $\dot{\mathcal{G}}^\lambda$, $\lambda \in \Lambda$, is a subalgebra of $\dot{\mathcal{L}}$. Also $\dot{\tau} : \dot{\mathcal{L}} \times \dot{\mathcal{L}} \longrightarrow ker(\pi)$ is a 2-cocycle. Moreover consulting Lemma 1.36, we get that $\dot{\mathcal{L}}$ is a $\mathcal{G}$−submodule of $\tilde{\mathcal{L}}$ which implies that

(1.41) $$\dot{\tau}(\dot{\mathcal{G}}, \dot{\mathcal{L}}) = \{0\}.$$

For each $x \in \tilde{\mathcal{L}}$, there are unique $\dot{\ell}_x \in \dot{\mathcal{L}}$, $\ell_x \in \mathcal{L}$ and $e_x, f_x \in ker(\pi)$ such that $x = \dot{\ell}_x + e_x = \ell_x + f_x$. Also we note that

$$(1.42) \qquad \begin{array}{llll} \ell_{\dot{\ell}_x} = x & \text{and} & f_{\dot{\ell}_x} = -e_x; & x \in \mathcal{L} \\ \dot{\ell}_{\ell_y} = y & \text{and} & e_{\ell_y} = -f_y; & y \in \dot{\mathcal{L}}, \end{array}$$

(see (1.35)). For $\lambda \in \Lambda$, set

$$\dot{\mathcal{L}}^\lambda := \sum_{i \in \mathcal{I}} \dot{\mathcal{G}}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \dot{\mathcal{S}}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \dot{\mathcal{V}}_t^\lambda \oplus \dot{\mathcal{D}}.$$

So considering (1.20), we have

$$(1.43) \qquad \tilde{\mathcal{L}}^\lambda := \mathcal{L}^\lambda \oplus ker(\pi) = \dot{\mathcal{L}}^\lambda \oplus ker(\pi).$$

Note that for each $\lambda \in \Lambda$, $\tilde{\mathcal{L}}^\lambda$ is a Lie subalgebra of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}$ is the direct union of $\{\tilde{\mathcal{L}}^\lambda \mid \lambda \in \Lambda\}$. In the following, we show that for $\lambda \in \Lambda$, $\dot{\mathcal{L}}^\lambda$ is a Lie subalgebra of $\dot{\mathcal{L}}$ and that $\dot{\mathcal{L}}$ is the direct union of $\{\dot{\mathcal{L}}^\lambda \mid \lambda \in \Lambda\}$.

Lemma 1.44. *(i)* $\pi|_{\dot{\mathcal{L}}}$ *is a Lie algebra isomorphism from* $(\dot{\mathcal{L}}, [\cdot, \cdot])$ *to* $(\mathcal{L}, [\cdot, \cdot])$. *Also for each* $\lambda \in \Lambda$, $\dot{\mathcal{L}}^\lambda$ *is a Lie subalgebra of* $(\dot{\mathcal{L}}, [\cdot, \cdot])$ *isomorphic to* $\mathcal{L}^\lambda$ *and* $\dot{\mathcal{L}}$ *is the direct union of* $\{\dot{\mathcal{L}}^\lambda \mid \lambda \in \Lambda\}$.

(ii) Recall (1.39), for $\lambda \in \Lambda$, $\pi_1|_{\tilde{\mathcal{L}}^\lambda} : \tilde{\mathcal{L}}^\lambda \longrightarrow \dot{\mathcal{L}}^\lambda$ *is a central extension of* $\dot{\mathcal{L}}^\lambda$ *with corresponding 2-cocycle* $\dot{\tau}\,|_{\dot{\mathcal{L}}^\lambda \times \dot{\mathcal{L}}^\lambda}$ *satisfying* $\dot{\tau}(\dot{\mathcal{G}}^\lambda, \dot{\mathcal{L}}^\lambda) = \{0\}$.

(iii) For $\lambda \in \Lambda$, *there is a subalgebra* $\dot{\mathcal{D}}_\lambda$ *of* $\dot{\mathcal{L}}^\lambda$ *with* $\pi(\dot{\mathcal{D}}_\lambda) = \mathcal{D}_\lambda$ *such that* $\dot{\mathcal{D}}_\lambda$ *is a trivial* $\dot{\mathcal{G}}^\lambda$−*submodule of* $\dot{\mathcal{L}}^\lambda$ *and*

$$\begin{aligned} \dot{\mathcal{L}}^\lambda &= \sum_{i \in \mathcal{I}} \dot{\mathcal{G}}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \dot{\mathcal{S}}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \dot{\mathcal{V}}_t^\lambda \oplus \dot{\mathcal{D}} \\ &= \sum_{i \in \mathcal{I}} \dot{\mathcal{G}}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \dot{\mathcal{S}}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \dot{\mathcal{V}}_t^\lambda \oplus \dot{\mathcal{D}}_\lambda. \end{aligned}$$

Proof. *(i)* We fix $x, y \in \dot{\mathcal{L}}$ and show that $\pi([x, y]) = [\pi(x), \pi(y)] = [\ell_x, \ell_y]$. Using (1.42), we have

$$\begin{aligned} \pi([x, y]) = \pi(\pi_1([x, \tilde{y}])) &= \pi(\pi_1([\ell_x, \ell_y] + \tau(\ell_x, \ell_y))) \\ &= \pi(\pi_1(\dot{\ell}_{[\ell_x, \ell_y]} + e_{[\ell_x, \ell_y]} + \tau(\ell_x, \ell_y))) \\ &= \pi(\dot{\ell}_{[\ell_x, \ell_y]}) \\ &= \ell_{\dot{\ell}_{[\ell_x, \ell_y]}} = [\ell_x, \ell_y]. \end{aligned}$$

This means that the restriction of π to $\dot{\mathcal{L}}$ is a Lie algebra homomorphism. But $\tilde{\mathcal{L}} = \mathcal{L} \oplus ker(\pi) = \dot{\mathcal{L}} \oplus ker(\pi)$ which in turn implies that π restricted to $\dot{\mathcal{L}}$ is an isomorphism from $\dot{\mathcal{L}}$ onto $\mathcal{L}$. Next suppose $\lambda \in \Lambda$ and $x, y \in \dot{\mathcal{L}}^\lambda$, then $\ell_x, \ell_y \in \mathcal{L}^\lambda$. Also $[x, y] \in \dot{\mathcal{L}}$, and $[x, y] = [x, \tilde{y}] - \dot{\tau}(x, y) = [\ell_x, \ell_y] + \tau(\ell_x, \ell_y) - \dot{\tau}(x, y) \in \mathcal{L}^\lambda + ker(\pi) = \tilde{\mathcal{L}}^\lambda$. Therefore we get $[x, y] \in \dot{\mathcal{L}} \cap \tilde{\mathcal{L}}^\lambda = \dot{\mathcal{L}}^\lambda$ which shows that $\dot{\mathcal{L}}^\lambda$ is a subalgebra of $\dot{\mathcal{L}}$. Now as the restriction of π to $\dot{\mathcal{L}}$ is a Lie algebra isomorphism from $\dot{\mathcal{L}}$ to $\mathcal{L}$, we get using (1.43) that for $\lambda \in \Lambda$,

the restriction of π to $\dot{\mathcal{L}}^\lambda$ is a Lie algebra isomorphism from $\dot{\mathcal{L}}^\lambda$ to $\mathcal{L}^\lambda$. Now consider (1.19) and set $\dot{\mathcal{D}}_\lambda := \dot{\mathcal{L}}^\lambda \cap \pi^{-1}(\mathcal{D}_\lambda)$. We note that $\pi : \dot{\mathcal{L}}^\lambda \longrightarrow \mathcal{L}^\lambda$ is a central extension, so using Lemma 1.15, we have $[\dot{\mathcal{G}}^\lambda, \dot{\mathcal{D}}_\lambda\tilde{]} = [\mathcal{G}^\lambda, \mathcal{D}_\lambda\tilde{]} = 0$.

(ii) Since $\tilde{\mathcal{L}}^\lambda = \dot{\mathcal{L}}^\lambda \oplus ker(\pi)$, it is immediate that $\pi_1|_{\tilde{\mathcal{L}}^\lambda}$ is a central extension of $\dot{\mathcal{L}}^\lambda$. Also we note that for $x \in \dot{\mathcal{G}}^\lambda$ and $y \in \dot{\mathcal{L}}^\lambda$, $[\ell_x, y\tilde{]} \in \dot{\mathcal{L}}^\lambda$ as $\dot{\mathcal{L}}^\lambda$ is a $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}$. Thus $\dot{\tau}(x, y) = \pi_2([x, y\tilde{]}) = \pi_2([\ell_x, y\tilde{]}) = 0$. □

Now using the same notation as in the text and regard Lemmas 1.36 and 1.44, we summarize our information as follows: We have

$$\dot{\mathcal{D}} \subseteq [\mathcal{L}^0, \mathcal{L}^0\tilde{]} = [\dot{\mathcal{L}}^0, \dot{\mathcal{L}}^0\tilde{]},$$

$$\tilde{\mathcal{L}} = \dot{\mathcal{L}} \oplus ker(\pi) = \sum_{i \in \mathcal{I}} \dot{\mathcal{G}}_i \oplus \sum_{j \in \mathcal{J}} \dot{\mathcal{S}}_j \oplus \sum_{t \in \mathcal{T}} \dot{\mathcal{V}}_t \oplus \dot{\mathcal{D}} \oplus ker(\pi),$$

$$\tilde{\mathcal{L}}^\lambda = \dot{\mathcal{L}}^\lambda \oplus ker(\pi) \quad = \sum_{i \in \mathcal{I}} \dot{\mathcal{G}}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \dot{\mathcal{S}}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \dot{\mathcal{V}}_t^\lambda \oplus \dot{\mathcal{D}} \oplus ker(\pi)$$
$$= \sum_{i \in \mathcal{I}} \dot{\mathcal{G}}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \dot{\mathcal{S}}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \dot{\mathcal{V}}_t^\lambda \oplus \dot{\mathcal{D}}_\lambda \oplus ker(\pi),$$

($\lambda \in \Lambda$). Also $\pi_1 : \tilde{\mathcal{L}} \longrightarrow \dot{\mathcal{L}}$ is a central extension of $\dot{\mathcal{L}}$ with corresponding 2-cocycle $\dot{\tau}$ satisfying $\dot{\tau}(\dot{\mathcal{G}}, \dot{\mathcal{L}}) = \{0\}$. So without loss of generality, from now on we assume that $\tilde{\mathcal{L}} = \mathcal{L} \oplus ker(\pi)$ and that $\mathcal{L}$ is a $\mathcal{G}$−submodule of $\tilde{\mathcal{L}}$ and that

(1.45) $$\tau(\mathcal{G}, \mathcal{L}) = \{0\} \quad \text{and} \quad \mathcal{D}_0 \subseteq [\mathcal{L}^0, \mathcal{L}^0\tilde{]} = [\tilde{\mathcal{L}}^0, \tilde{\mathcal{L}}^0\tilde{]}.$$

Lemma 1.46. *For $\lambda \in \Lambda$, $\mathcal{L}^\lambda \subseteq [\mathcal{L}^\lambda, \mathcal{L}^\lambda\tilde{]}$. Also $\mathcal{L} \subseteq [\mathcal{L}, \mathcal{L}\tilde{]}$.*

Proof. We know from (1.19) that
(1.47)
$$\mathcal{L}^\lambda = (\mathcal{G}^\lambda \dot{\otimes} \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot{\otimes} \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot{\otimes} \mathcal{C}) \oplus \mathcal{D}_\lambda = (\mathcal{G}^\lambda \dot{\otimes} \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot{\otimes} \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot{\otimes} \mathcal{C}) \oplus \mathcal{D}_0.$$
and that $\mathcal{L} = (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \mathcal{D}_0$. We also know that the restriction π to $\mathcal{L}^\lambda \oplus ker(\pi)$ is a central extension of $\mathcal{L}^\lambda$ with corresponding 2-cocycle τ satisfying $\tau(\mathcal{G}^\lambda, \mathcal{L}^\lambda) = \{0\}$. Thus it follows from [2, Pro. 5.23] that the summands $\mathcal{G}^\lambda \otimes \mathcal{A}$, $\mathcal{S}^\lambda \otimes \mathcal{B}$, $\mathcal{V}^\lambda \otimes \mathcal{C}$ and $\mathcal{D}^\lambda$ are orthogonal with respect to τ and that for $x, y \in \mathcal{G}^\lambda$, $a \in \mathcal{A}$, $s \in \mathcal{S}$, $b \in \mathcal{B}$, $v \in \mathcal{V}$ and $c \in \mathcal{C}$, we have

$$[x \otimes 1, y \otimes a\tilde{]} = [x \otimes 1, y \otimes a] + \tau(x \otimes 1, y \otimes a) = [x \otimes 1, y \otimes a] = [x, y] \otimes a,$$
$$[x \otimes 1, s \otimes b\tilde{]} = [x \otimes 1, s \otimes b] = [x, s] \otimes b,$$
$$[x \otimes 1, v \otimes c\tilde{]} = [x \otimes 1, v \otimes c] = xv \otimes c.$$

This together with the fact that $\mathcal{G}, \mathcal{S}$ and $\mathcal{V}$ are irreducible finite dimensional $\mathcal{G}$−modules, implies that $(\mathcal{G}^\lambda \dot{\otimes} \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot{\otimes} \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot{\otimes} \mathcal{C}) \subseteq [\tilde{\mathcal{L}}^\lambda, \tilde{\mathcal{L}}^\lambda\tilde{]}$. Now contemplating (1.47) and (1.45), we are done. □

Theorem 1.48. *Suppose that I is an infinite index set, R is an irreducible locally finite root system of type BC_I and $\mathfrak{q} := (\mathfrak{a}, *, \mathcal{C}, f)$ is a coordinate quadruple of type BC. Take $\mathfrak{b} := \mathfrak{b}(\mathfrak{q})$ and suppose $\mathcal{K}$ is a subspace of $\mathrm{HF}(\mathfrak{b})$ satisfying the uniform property on $\mathfrak{b}$. Set $\langle \mathfrak{b}, \mathfrak{b} \rangle := \{\mathfrak{b}, \mathfrak{b}\}/\mathcal{K}$ and consider the*

$R-$graded Lie algebra $\mathcal{L} := (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \langle \mathfrak{b}, \mathfrak{b} \rangle$. Suppose that $\tau : \mathcal{L} \times \mathcal{L} \longrightarrow E$ is a 2-cocycle and consider the corresponding central extension $\tilde{\mathcal{L}} := \mathcal{L} \oplus E$ as well as the canonical projection map $\pi : \tilde{\mathcal{L}} \longrightarrow \mathcal{L}$. If $\tilde{\mathcal{L}}$ is perfect, then $\tilde{\mathcal{L}}$ is an $R-$graded Lie algebra with the same coordinate quadruple $(\mathfrak{a}, *, \mathcal{C}, f)$. Also there is a subspace $\mathcal{K}_0$ of $\mathrm{HF}(\mathfrak{b})$ satisfying the uniform property on $\mathfrak{b}$ such that $\tilde{\mathcal{L}}$ can be identified with $(\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus (\{\mathfrak{b}, \mathfrak{b}\}/\mathcal{K}_0)$ where setting $\langle b, b' \rangle_c := \{b, b'\} + \mathcal{K}_0$, the Lie bracket on $\tilde{\mathcal{L}}$ is given by

(1.49)

$$[\widetilde{x \otimes a, y \otimes a'}] = [x, y] \otimes \tfrac{1}{2}(a \circ a') + (x \circ y) \otimes \tfrac{1}{2}[a, a'] + tr(xy)\langle a, a' \rangle_c,$$

$$[\widetilde{x \otimes a, s \otimes b}] = (x \circ s) \otimes \tfrac{1}{2}[a, b] + [x, s] \otimes \tfrac{1}{2}(a \circ b) = -[s \otimes b, x \otimes a],$$

$$[\widetilde{s \otimes b, t \otimes b'}] = [s, t] \otimes \tfrac{1}{2}(b \circ b') + (s \circ t) \otimes \tfrac{1}{2}[b, b'] + tr(st)\langle b, b' \rangle_c,$$

$$[\widetilde{x \otimes a, u \otimes c}] = xu \otimes a \cdot c = -[u \otimes c, x \otimes a],$$

$$[\widetilde{s \otimes b, u \otimes c}] = su \otimes b \cdot c = -[u \otimes c, s \otimes b],$$

$$[\widetilde{u \otimes c, v \otimes c'}] = (u \circ v) \otimes (c \diamond c') + [u, v] \otimes (c \heartsuit c') + (u, v)\langle c, c' \rangle_c,$$

$$[\widetilde{\langle \beta_1, \beta_2 \rangle, x \otimes a}] = \tfrac{-1}{4\ell}(x \circ Id_{\mathcal{V}\ell} \otimes [a, \beta_{\beta_1,\beta_2}^*] + [x, Id_{\mathcal{V}\ell}] \otimes a \circ \beta_{\beta_1,\beta_2}^*),$$

$$[\widetilde{\langle \beta_1, \beta_2 \rangle, s \otimes b}] = \tfrac{-1}{4\ell}([s, Id_{\mathcal{V}\ell}] \otimes (b \circ \beta_{\beta_1,\beta_2}^*) + (s \circ Id_{\mathcal{V}\ell}) \otimes [b, \beta_{\beta_1,\beta_2}^*] + 2tr(s Id_{\mathcal{V}\ell})\langle b, \beta_{\beta_1,\beta_2}^* \rangle_c),$$

$$[\widetilde{\langle \beta_1, \beta_2 \rangle_c, v \otimes c}] = \tfrac{1}{2\ell} Id_{\mathcal{V}\ell} v \otimes (\beta_{\beta_1,\beta_2}^* \cdot c) - \tfrac{1}{2} v \otimes (f(c, \beta_2^*) \cdot \beta_1^* + f(c, \beta_1^*) \cdot \beta_2^*)$$

$$[\widetilde{\langle \beta_1, \beta_2 \rangle_c, \langle \beta_1', \beta_2' \rangle_c}] = \langle d_{\beta_1,\beta_2}^\ell(\beta_1'), \beta_2' \rangle_c + \langle \beta_1', d_{\beta_1,\beta_2}^\ell(\beta_2') \rangle_c$$

for $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $u, v \in \mathcal{V}$, $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $c, c' \in \mathcal{C}$, $\beta_1, \beta_2, \beta_1', \beta_2' \in \mathfrak{b}$. Moreover, under the above identification, $\pi : \tilde{\mathcal{L}} \longrightarrow \mathcal{L}$ is given by $\pi(x) = x$ for $x \in (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C})$ and $\pi(\langle b, b' \rangle) = \langle b, b' \rangle_c$ for $b, b' \in \mathfrak{b}$.

Proof. As we have already seen, without loss of generality, we may assume $\tau(\mathcal{G}, \mathcal{L}) = \{0\}$. We now note that $\tilde{\mathcal{L}}$ is an $R-$graded Lie algebra with grading pair $(\mathcal{G}, \mathcal{H})$ and weight space decomposition $\tilde{\mathcal{L}} = \oplus_{\alpha \in R} \tilde{\mathcal{L}}_\alpha$ where

(1.50) $$\tilde{\mathcal{L}}_\alpha = \mathcal{L}_\alpha; \ \alpha \in R \setminus \{0\}, \ \tilde{\mathcal{L}}_0 = \mathcal{L}_0 \oplus ker(\pi) = \mathcal{L}_0 \oplus E.$$

Suppose that $\{a_i \mid i \in \mathcal{I}\}$, $\{b_j \mid j \in \mathcal{J}\}$ and $\{c_t \mid t \in \mathcal{T}\}$ are bases for $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ respectively. We assume $0 \in \mathcal{I}$ and $a_0 = 1$. For $\lambda \in \Lambda$ and $i \in \mathcal{I}, j \in \mathcal{J}$ and $t \in \mathcal{T}$, we set

$$\mathcal{G}_i^\lambda := \mathcal{G}^\lambda \otimes a_i, \ \mathcal{G}_i := \mathcal{G} \otimes a_i$$
$$\mathcal{S}_j^\lambda := \mathcal{S}^\lambda \otimes b_j, \ \mathcal{S}_j := \mathcal{S} \otimes b_j$$
$$\mathcal{V}_t^\lambda := \mathcal{V}^\lambda \otimes c_t, \ \mathcal{V}_t := \mathcal{V} \otimes c_t.$$

Therefore for $\mathcal{D} := \langle \mathfrak{b}, \mathfrak{b} \rangle$, we have $\mathcal{L} = \sum_{i \in \mathcal{I}} \mathcal{G}_i \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t \oplus \mathcal{D}$, $\mathcal{G}_i = \cup_{\lambda \in \Lambda} \mathcal{G}_i^\lambda$, $\mathcal{S}_j = \cup_{\lambda \in \Lambda} \mathcal{S}_j^\lambda$ and $\mathcal{V}_t = \cup_{\lambda \in \Lambda} \mathcal{V}_t^\lambda$, $i \in \mathcal{I}, j \in \mathcal{J}, t \in \mathcal{T}$. For

$\lambda \in \lambda$, set

$$\mathcal{L}^\lambda := \sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t^\lambda \oplus \mathcal{D},$$

$$\hat{\mathcal{L}}^\lambda := \sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t^\lambda \oplus \mathcal{D} \oplus ker(\pi),$$

$$\tilde{\mathcal{L}}^\lambda := [\hat{\mathcal{L}}^\lambda, \hat{\mathcal{L}}^\lambda\tilde{]} = [\mathcal{L}^\lambda, \mathcal{L}^\lambda\tilde{]}.$$

The restriction of π to $\hat{\mathcal{L}}^\lambda$ is a central extension of $\mathcal{L}^\lambda$ and setting $\pi_\lambda := \pi|_{\tilde{\mathcal{L}}^\lambda} : \tilde{\mathcal{L}}^\lambda \longrightarrow \mathcal{L}^\lambda$, we get that $(\tilde{\mathcal{L}}^\lambda, \pi_\lambda)$ is a perfect central extension of $\mathcal{L}^\lambda$. Also by Lemma 1.46, we have

(1.51) $$\tilde{\mathcal{L}}^\lambda = \mathcal{L}^\lambda \oplus \mathcal{Z}_\lambda$$

where $\mathcal{Z}_\lambda := ker(\pi_\lambda)$. Now as $\tilde{\mathcal{L}}$ is perfect,

$$\tilde{\mathcal{L}} = [\tilde{\mathcal{L}}, \tilde{\mathcal{L}}\tilde{]} = [\cup_{\lambda \in \Lambda}\hat{\mathcal{L}}^\lambda, \cup_{\lambda \in \Lambda}\hat{\mathcal{L}}^\lambda] = \cup_{\lambda \in \Lambda}[\hat{\mathcal{L}}^\lambda, \hat{\mathcal{L}}^\lambda] = \cup_{\lambda \in \Lambda}\tilde{\mathcal{L}}^\lambda$$

so $\tilde{\mathcal{L}}$ is the direct union of $\{\tilde{\mathcal{L}}^\lambda \mid \lambda \in \Lambda\}$. We next note that $\mathcal{L}^\lambda$ is an R_λ−graded Lie algebra with grading pair $(\mathcal{G}^\lambda, \mathcal{H}_\lambda := \mathcal{G}^\lambda \cap \mathcal{H})$ and $\tilde{\mathcal{L}}^\lambda$ is a perfect central extension of $\mathcal{L}^\lambda$ with corresponding 2−cocycle $\tau_\lambda := \tau \mid_{\mathcal{L}^\lambda \times \mathcal{L}^\lambda}$ satisfying $\tau_\lambda(\mathcal{G}^\lambda, \mathcal{L}^\lambda) = \{0\}$. Therefore by Lemma 1.16, $\tilde{\mathcal{L}}^\lambda = \oplus_{\alpha \in R_\lambda}\tilde{\mathcal{L}}_\alpha^\lambda$ with

$$\tilde{\mathcal{L}}_\alpha^\lambda = \left\{ \begin{array}{ll} \mathcal{L}_\alpha^\lambda = \mathcal{L}_\alpha & \text{if } \alpha \in R_\lambda \setminus \{0\} \\ \mathcal{L}_0^\lambda \oplus \mathcal{Z}_\lambda & \text{if } \alpha = 0 \end{array} \right. = \left\{ \begin{array}{ll} \mathcal{L}_\alpha & \text{if } \alpha \in R_\lambda \setminus \{0\} \\ \displaystyle\sum_{\beta \in R_\lambda \setminus \{0\}} [\mathcal{L}_\beta, \mathcal{L}_{-\beta}\tilde{]} & \text{if } \alpha = 0. \end{array} \right.$$

We next recall from (1.51) that for $\lambda \in \Lambda$, $\tilde{\mathcal{L}}^\lambda = \sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t^\lambda \oplus \mathcal{D}_\lambda \oplus \mathcal{Z}_\lambda$. Also $\mathcal{D}_\lambda \oplus \mathcal{Z}_\lambda$ is a trivial $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}^\lambda$. We next note that $\tau(\mathcal{G}, \mathcal{L}) = \{0\}$ which implies that $\mathcal{L}^\lambda$ is a $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}^\lambda$. Now as $\mathcal{G}_i^\lambda$ is the $\mathcal{G}^\lambda$−submodule of $\mathcal{L}^\lambda$ generated by $\mathcal{G}_i^0$, we get that $\mathcal{G}_i^\lambda$ is also the $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}^\lambda$ generated by $\mathcal{G}_i^0$. Similarly for $j \in \mathcal{J}$ and $t \in \mathcal{T}$, $\mathcal{S}_j^\lambda$ coincides with the $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}^\lambda$ generated by $\mathcal{S}_j^0$, and $\mathcal{V}_t^\lambda$ coincides with the $\mathcal{G}^\lambda$−submodule of $\tilde{\mathcal{L}}^\lambda$ generated by $\mathcal{V}_t^0$. This means that

$$(\mathcal{I}, \mathcal{J}, \mathcal{T}, \{\mathcal{G}_i^0\}, \{\mathcal{G}_i^\lambda\}, \{\mathcal{S}_j^0\}, \{\mathcal{S}_j^\lambda\}, \{\mathcal{V}_t^0\}, \{\mathcal{V}_t^\lambda\}, \mathcal{D}_0 \oplus \mathcal{Z}_0, \mathcal{D}_\lambda \oplus \mathcal{Z}_\lambda)$$

is an (R_0, R_λ)−datum for $0 \prec \lambda$ in the sense of [8]. Therefore using [Y], we get that

$$\begin{aligned} \tilde{\mathcal{L}}^\lambda &= \sum_{i \in \mathcal{I}} \mathcal{G}_i^\lambda \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j^\lambda \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t^\lambda \oplus \mathcal{D}_0 \oplus \mathcal{Z}_0 \\ &= (\mathcal{G}^\lambda \dot\otimes \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot\otimes \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot\otimes \mathcal{C}) \oplus \mathcal{D}_0 \oplus \mathcal{Z}_0. \end{aligned}$$

So

$$\begin{aligned} \tilde{\mathcal{L}} = \cup \tilde{\mathcal{L}}^\lambda &= \sum_{i \in \mathcal{I}} \mathcal{G}_i \oplus \sum_{j \in \mathcal{J}} \mathcal{S}_j \oplus \sum_{t \in \mathcal{T}} \mathcal{V}_t \oplus \mathcal{D}_0 \oplus \mathcal{Z}_0 \\ &= (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \mathcal{D}_0 \oplus \mathcal{Z}_0. \end{aligned}$$

Using [Y] and Lemma 1.16, $\mathcal{L}^0$, $\tilde{\mathcal{L}}^0$, $\tilde{\mathcal{L}}^\lambda$ and $\mathcal{L}^\lambda$ have the same coordinate quadruple $\mathfrak{q}$. Also there is a subspace $\mathcal{K}_0$ of $\mathrm{HF}(\mathfrak{b})$ satisfying the uniform property on $\mathfrak{b}$ such that $\mathcal{D}_0 \oplus \mathcal{Z}_0 = \{\mathfrak{b}, \mathfrak{b}\}/\mathcal{K}_0$. Now setting $\langle \mathfrak{b}, \mathfrak{b} \rangle_c :=$ $\{\mathfrak{b}, \mathfrak{b}\}/\mathcal{K}_0$, we get using (1.49) using [Y, Pro 3.10]. Now by [Y, Theorem 4.1], $\tilde{\mathcal{L}}$ is an $R-$graded Lie algebra. Now for fix $x, y \in \mathcal{G}$ with $tr(xy) \neq 0$, and $a, a' \in \mathcal{A}$, we have

$$
\begin{aligned}
& [x, y] \otimes (1/2)(a \circ a') + (x \circ y) \otimes (1/2)[a, a'] + tr(xy)\langle a, a' \rangle \\
=\ & [x \otimes a, y \otimes a'] \\
=\ & [\pi(x \otimes a), \pi(y \otimes a')] \\
=\ & \pi([x \otimes a, y \otimes \tilde{a'}]) \\
=\ & \pi([x, y] \otimes (1/2)(a \circ a') + (x \circ y) \otimes (1/2)[a, a'] + tr(xy)\langle a, a' \rangle_c) \\
=\ & [x, y] \otimes (1/2)(a \circ a') + (x \circ y) \otimes (1/2)[a, a'] + tr(xy)\pi(\langle a, a' \rangle_c).
\end{aligned}
$$

This implies that $\pi(\langle a, a' \rangle_c) = \langle a, a' \rangle$. Similarly we can prove that $\pi(\langle b, b' \rangle_c) = \langle b, b' \rangle$ and $\pi(\langle c, c' \rangle_c) = \langle c, c' \rangle$ for $b, b' \in \mathcal{B}$ and $c, c' \in \mathcal{C}$. This completes the proof. $\qquad \square$

Theorem 1.52. *Suppose that $\mathfrak{q} := (\mathfrak{a}, *, \mathcal{C}, f)$ is a coordinate quadruple of type BC, $\mathfrak{b} := \mathfrak{b}(\mathfrak{q})$, $\mathcal{K}$ a subspace of $\mathrm{HF}(\mathfrak{b})$ satisfying the uniform property on $\mathfrak{b}$ and $\mathcal{L} := \mathcal{L}(\mathfrak{q}, \mathcal{K}) = (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \langle \mathfrak{b}, \mathfrak{b} \rangle$, where $\langle \mathfrak{b}, \mathfrak{b} \rangle :=$ $\{\mathfrak{b}, \mathfrak{b}\}/\mathcal{K}$, the corresponding $R-$graded Lie algebra. Consider Remark 1.7 and set $\mathfrak{A} := \mathcal{L}(\mathfrak{q}, \{0\}) = (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \{\mathfrak{b}, \mathfrak{b}\}$, then $\mathfrak{A}$ is the universal central extension of $\mathcal{L}$.*

Proof. Define

$$
\begin{aligned}
& \pi : \mathfrak{A} \longrightarrow \mathcal{L}; \\
& x \mapsto x; \quad x \in (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}); \\
& \{b, b'\}\rangle_u \mapsto \{b, b'\} + \mathcal{K} = \langle b, b' \rangle.
\end{aligned}
$$

If $x \in ker(\pi)$, then $x = \sum \{\beta_i, \beta_i'\}$ such that $\sum_i \langle \beta_i, \beta_i' \rangle = 0$. But since $\mathcal{K}$ satisfies the uniform property on $\mathfrak{b}$, we get that $\sum_i \beta^*_{\beta_i, \beta_i'} = 0$. Now (1.49) together with (1.14) implies that $x \in Z(\mathfrak{A})$. This means that π is a central extension of $\mathcal{L}$. Now suppose that $\dot{\mathcal{L}}$ is a Lie algebra and $\dot{\varphi} : \dot{\mathcal{L}} \longrightarrow \mathcal{L}$ is a central extension of $\mathcal{L}$. Set $\tilde{\mathcal{L}}$ to be the derived algebra of $\dot{\mathcal{L}}$ and $\varphi := \dot{\varphi}\mid_{\tilde{\mathcal{L}}}$. Then $(\tilde{\mathcal{L}}, \varphi)$ is a perfect central extension of $\mathcal{L}$. By Theorem 1.48, we may assume there is a subspace $\mathcal{K}_0$ of $\mathrm{HF}(\mathfrak{b})$ satisfying the uniform property on $\mathfrak{b}$ such that $\tilde{\mathcal{L}} = (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \langle \mathfrak{b}, \mathfrak{b} \rangle_c$ where $\langle \mathfrak{b}, \mathfrak{b} \rangle_c := \{\mathfrak{b}, \mathfrak{b}\}/\mathcal{K}_0$ and $\varphi : \tilde{\mathcal{L}} \longrightarrow \mathcal{L}$ is given by

$$
\begin{aligned}
& \varphi(x) = x; \quad x \in (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \\
& \varphi(\langle \beta, \beta' \rangle_c) = \langle \beta, \beta' \rangle; \quad \beta, \beta' \in \mathfrak{b}.
\end{aligned}
$$

Now if we define

$$
\begin{aligned}
& \psi : (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \{\mathfrak{b}, \mathfrak{b}\} \longrightarrow \tilde{\mathcal{L}} \\
& \psi(x) = x; \quad x \in (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \quad \text{and} \quad \psi(\{\beta, \beta'\}) = \langle \beta, \beta' \rangle_c,
\end{aligned}
$$

ψ is a Lie algebra homomorphism satisfying $\varphi \circ \psi = \pi$. In other words, π is the universal central extension. $\qquad\square$

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